ارتعاشات غیر خطی

Perturbation theory, Multiple Scale Method

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Introduction to perturbation techniques, A.H. Nayfeh, chapter 6

straight forward expantion درجلسه قبل روش

$$\ddot{x} + 2\varepsilon\dot{x} + x = 0$$

$$x(0) = 0, \quad \dot{x}(0) = 1, \quad \varepsilon \ll 1$$

$$x(t) = x_0(t) + \varepsilon x_1(t) + \varepsilon^2 x_2(t) + O(\varepsilon^3) + \cdots$$

$$\varepsilon^0: \ddot{x}_0 + x_0 = 0 \longrightarrow x_0 = a\cos(t+\varphi)$$

$$\varepsilon^1: \ddot{x}_1 + x_1 = -2\dot{x}_0 \longrightarrow x_1 = -at\cos(t+\varphi)$$

$$\varepsilon^2: \ddot{x}_2 + x_2 = -2\dot{x}_1 \longrightarrow x_2 = \frac{1}{2}at^2\cos(t+\varphi) + \frac{1}{2}at\sin(t+\varphi)$$

$$t \ll O(\varepsilon^{-2})$$

Exact sol.: $x(t) = a e^{-\varepsilon t} \cos\left(\sqrt{1-\varepsilon^2}\, t + \varphi\right)$

$$\sqrt{1-\varepsilon^2} = 1 - \frac{\varepsilon^2}{2} + \frac{\varepsilon^4}{8} - \cdots$$

$$x(t) = a e^{-\varepsilon t} \cos\left(t - \frac{\varepsilon^2 t}{2} + \cdots + \varphi\right) = x(T_0, T_1, T_2)$$

$\varepsilon t \rightarrow T_1$, $t \rightarrow T_0$, $\varepsilon^2 t \rightarrow T_2$

با تعریف اسکیل های زمانی مختلف

$T_0 = t \longrightarrow$ fast (second)

$T_1 = \varepsilon t = O(1) \Rightarrow t = O(\varepsilon^{-1}) \longrightarrow$ slow (minute)

$T_2 = \varepsilon^2 t \Rightarrow t = O(\varepsilon^{-2}) \longrightarrow$ very slow (hour)

$\vdots$

$T_n = \varepsilon^n t$

$$\Rightarrow x(t) = a e^{-T_1} \cos\left(T_0 - \frac{T_2}{2} + \varphi\right)$$

$$x(t,\varepsilon) = X_0(T_0,T_1,T_2) + \varepsilon X_1(T_0,T_1,T_2) + \varepsilon^2 X_2(T_0,T_1,T_2) + O(\varepsilon^3)$$

$$\dot{x} = \frac{dx}{dt} = \frac{\partial x}{\partial T_0}\underbrace{\frac{\partial T_0}{\partial t}}_{1} + \frac{\partial x}{\partial T_1}\underbrace{\frac{\partial T_1}{\partial t}}_{\varepsilon} + \frac{\partial x}{\partial T_2}\underbrace{\frac{\partial T_2}{\partial t}}_{\varepsilon^2}$$

$$x(T_0,T_1,T_2), \quad T_0 = t, \quad T_1 = \varepsilon t, \quad T_2 = \varepsilon^2 t$$

$$\frac{dx}{dt} = \dot{x} = \frac{\partial x}{\partial T_0} + \varepsilon \frac{\partial x}{\partial T_1} + \varepsilon^2 \frac{\partial x}{\partial T_2} = D_0 x + \varepsilon D_1 x + \varepsilon^2 D_2 x$$

$$\frac{\partial}{\partial T_0} = D_0, \quad \frac{\partial}{\partial T_1} = D_1, \quad \frac{\partial}{\partial T_2} = D_2, \quad \frac{\partial}{\partial T_n} = D_n$$

$$\Rightarrow \boxed{\frac{d}{dt} = D_0 + \varepsilon D_1 + \varepsilon^2 D_2}$$

$$\ddot{x} = \frac{d}{dt}(\dot{x}) = (D_0 + \varepsilon D_1 + \varepsilon^2 D_2)(D_0 x + \varepsilon D_1 x + \varepsilon^2 D_2 x)$$

$$= D_0^2 x + \varepsilon D_0 D_1 x + \varepsilon^2 D_0 D_2 x + \varepsilon D_1 D_0 x + \varepsilon^2 D_1^2 x + \varepsilon^3 D_1 D_2 x + \varepsilon^2 D_2 D_0 x$$
$$+ \varepsilon^3 D_2 D_1 x + \varepsilon^4 D_2^2 x$$

نکته: $D_0 D_1 x = \frac{\partial}{\partial T_0}\left(\frac{\partial}{\partial T_1} x\right) = D_1 D_0 x$

$$\Rightarrow \ddot{x} = D_0^2 x + 2\varepsilon D_0 D_1 x + \varepsilon^2 (D_1^2 + 2 D_0 D_2) x + \cdots$$

$$\Rightarrow \boxed{\frac{d^2}{dt^2} = D_0^2 + 2\varepsilon D_0 D_1 + \varepsilon^2 (D_1^2 + 2 D_0 D_2)}$$

EX: $\ddot{x} + 2\varepsilon\dot{x} + x = 0$, $x(t) = \underline{X_0(T_0, T_1)} + \varepsilon \underline{X_1(T_0, T_1)}$

$\dot{x} = (D_0 + \varepsilon D_1)[X_0 + \varepsilon X_1] = [D_0 X_0 + \varepsilon D_0 X_1 + \varepsilon D_1 X_0 + \varepsilon^2 D_1 X_1]\,2\varepsilon$

$\ddot{x} = (D_0^2 + 2\varepsilon D_0 D_1)[X_0 + \varepsilon X_1] = D_0^2 X_0 + \varepsilon D_0^2 X_1 + 2\varepsilon D_0 D_1 X_0 + \cdots$

$\varepsilon^0 : D_0^2 X_0 + X_0 = 0$ (1)

$\varepsilon^1 : D_0^2 X_1 + X_1 = -2 D_0 X_0 - 2 D_0 D_1 X_0$ (2)

(3)

تمرین $\varepsilon^2 : D_0^2 X_2 + X_2 = -2 D_0 D_1 X_1 - 2 D_0 D_2 X_0 - D_1^2 X_0 - 2 D_1 X_0 - 2 D_0 X_1$

و $x = X_0(T_0, T_1, T_2) + \varepsilon X_1 + \varepsilon^2 X_2$, ...

$$D_0^2 X_0 + X_0 = 0 \Rightarrow \frac{d^2 X_0}{dT_0^2} + X_0 = 0$$

شبیه ($\ddot{x} + x = 0$)

$$\frac{d^2}{dt^2}$$

$$\Rightarrow X_0(T_0, T_1, T_2) = a\cos(T_0 + \beta) = a(T_1, T_2)\cos(T_0 + \beta(T_1, T_2))$$

$$= A(T_1, T_2)e^{iT_0} + \bar{A}(T_1, T_2)e^{-iT_0}$$

$$= A(T_1, T_2)e^{iT_0} + cc$$ → complex congugate

$$\begin{cases} e^{i\theta} = \cos\theta + i\sin\theta \\ e^{-i\theta} = \cos\theta - i\sin\theta \end{cases} \Rightarrow \cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

② $D_0^2 X_1 + X_1 = -2D_0 X_0 - 2D_0 D_1 X_0$

$$-2D_0X_0 = -2\frac{\partial}{\partial T_0}\left(A(T_1,T_2)e^{iT_0} + cc\right) = -2iA(T_1,T_2)e^{iT_0} + cc$$

$$-2D_0D_1X_0 = -2\underbrace{D_1}_{\frac{\partial}{\partial T_1}}(D_0X_0) = -2D_1\left(\underline{iA(T_1,T_2)e^{iT_0} + cc}\right)$$

$$= -2i\underbrace{D_1A}_{\frac{\partial A}{\partial T_1}}\,e^{iT_0} + cc$$

$-2i(A + D_1A)\cos(T_0 + \beta)$

② $\Rightarrow D_0^2X_1 + X_1 = \overbrace{\underbrace{-2i(A + D_1A)e^{iT_0}}_{\text{secular term}} + cc}$

Eliminating secular term $\longrightarrow$ حذف $\Rightarrow D_0X_1 + X_1 = 0$

$\Rightarrow A + D_1A = 0 \Rightarrow A(T_1,T_2) = \beta(T_2)e^{-T_1}$

$\dot{A} = -A \Rightarrow A = e^{-t}$

جواب خصوصی $X_1 = 0$

$$\Rightarrow \quad X_0 = A(T_1, T_2)e^{iT_0} + cc$$

$$\Rightarrow \quad X_0 = B(T_2)e^{-T_1}e^{iT_0} + cc$$

$X_0, X_1 \longrightarrow$

$$③ \quad D_0^2 X_2 + X_2 = \underbrace{-2D_0D_1X_1}_{0} - 2D_0D_2X_0 - D_1^2X_0 - 2D_1X_0 \underbrace{-2D_0X_1}_{0} = Q$$

$$-2D_2D_0X_0 = -2D_2\left(iB(T_2)e^{-T_1}e^{iT_0} + cc\right)$$

$$= -2i(D_2B)e^{-T_1}e^{iT_0}$$

$$-D_1^2X_0 = -Be^{-T_1}e^{iT_0} + cc$$

$$-2D_1X_0 = 2Be^{-T_1}e^{iT_0} + cc$$

$$\Rightarrow Q = \left[-2iD_2B - B + 2B\right]e^{-T_1}e^{iT_0} + cc$$

$$Q = \left[-2iD_2 B - B + 2B\right] e^{-T_1} e^{iT_0} + cc \rightarrow \text{Secular term}$$

$$\Rightarrow -2iD_2 B(T_2) + B = 0 \Rightarrow -2iB' + B = 0 \Rightarrow B' + \frac{i}{2} B = 0$$

$$\Rightarrow B' = -\frac{i}{2} B \Rightarrow B(T_2) = c\, e^{-\frac{iT_2}{2}}$$

$$c = \frac{1}{2} a e^{i\varphi} \qquad \Rightarrow X_2 = 0$$

$$\Rightarrow B = \frac{1}{2} a e^{i\left(\varphi - \frac{T_2}{2}\right)}$$

$$X_0 = B(T_2) e^{-T_1} e^{iT_0} + cc = \frac{1}{2} a e^{-T_1} e^{i\left(T_0 - \frac{T_2}{2} + \varphi\right)} + cc$$

$$\Rightarrow X_0 = \frac{1}{2} a e^{-T_1} \cos\left(T_0 - \frac{T_2}{2} + \varphi\right)$$

$$X_0 = \frac{1}{2} a e^{-T_1} \cos\left(T_0 - \frac{T_2}{2} + \varphi\right)$$

$$\Rightarrow X_0 = \frac{1}{2} a \, e^{-\varepsilon t} \cos\left(t - \frac{\varepsilon^2 t}{2} + \varphi\right)$$

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