ارتعاشات غیر خطی

Perturbation theory، جواب عمومی و خصوصی

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در روش اغتشاشات گفتیم که:

در معادله اول ← هم جواب عمومی و هم خصوصی در نظر گرفته می‌شود

معادله دوم به بعد ← فقط جواب خصوصی در نظر گرفته می‌شود

چرا؟

در فصل دوم کتاب Nayfeh, nonlinear oscillations

این موضوع را با یک مثال توضیح داده است.

$\ddot{u} + f(u) = 0$, f is a nonlinear function

اگر نقطه تعادل $u=u_0$ باشد، با تعریف متغیر x داریم $\quad f(u_0)=0.$

$$x = u - u_0 \Rightarrow \ddot{x} = \ddot{u} - \underbrace{\ddot{u}_0}_{0} = \ddot{u} \Rightarrow \ddot{x} + f(x+u_0) = 0$$

$$f(x+u_0) = \sum_{n=1}^{N} \alpha_n x^n , \quad \alpha_n = \frac{f^{(n)}(u_0)}{n!}$$

$$\alpha_1 = \omega_0^2 = f'(u_0)$$

$$x(0) = s_0 = \varepsilon a_0 \cos\beta_0$$

$$\dot{x}(0) = v_0 = -\varepsilon a_0 \omega_0 \sin\beta_0$$

SFE با استفاده از روش

Nayfeh $\rightarrow$ $x(t,\varepsilon) = \varepsilon x_1 + \varepsilon^2 x_2 + \varepsilon^3 x_3 + \cdots =$

قبلاً $\rightarrow$ $x(t,\varepsilon) = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \cdots$

$$\ddot{x} + \omega_0^2 x + \alpha_2 x^2 + \alpha_3 x^3 = 0$$

$\varepsilon^1:$ $\ddot{x}_1 + \omega_0^2 x_1 = 0$ 1 $\rightarrow$ $x_1 = a\cos(\omega_0 t + \beta)$

$\varepsilon^2:$ $\ddot{x}_2 + \omega_0^2 x_2 = -\alpha_2 x_1^2$ 2

$\varepsilon^3:$ $\ddot{x}_3 + \omega_0^2 x_3 = -2\alpha_2 x_1 x_2 - \alpha_3 x_1^3$ 3

روش اول

شرایط اولیه

$$x_1(0) = a_0 \cos\beta_0 \quad , \quad \dot{x}_1(0) = -a_0\omega_0 \sin\beta_0 \qquad (4)$$

$$x_n(0) = 0 , \qquad \dot{x}_n(0) = 0 \qquad \text{for } n \geq 2 \qquad (5)$$

$$x_1 = a\cos(\omega_0 t + \beta) \overset{(4)}{\Longrightarrow} a = a_0 , \quad \beta = \beta_0 \Rightarrow x_1 = a_0 \cos(\omega_0 t + \beta_0)$$

$$\ddot{x}_2 + \omega_0^2 x_2 = -\alpha_2 a_0^2 \cos^2(\omega_0 t + \beta_0) = \frac{-\alpha_2 a_0^2}{2}\left[1 + \cos(2\omega_0 t + 2\beta_0)\right]$$

$$\left(\cos^2\varphi = \frac{1}{2}(1 + \cos 2\varphi)\right)$$

$$x_{2p} = A + B\cos(2\omega_0 t + 2\beta_0) \Rightarrow \dot{x}_{2p} = \checkmark \Rightarrow \ddot{x}_{2p} = \checkmark$$

$$\Rightarrow \omega_0^2 A = -\frac{1}{2}\alpha_2 a_0^2 , \quad A = \frac{-\alpha_2 a_0^2}{2\omega_0^2} , \quad B = \frac{-\alpha_2 a_0^2}{6\omega_0^2}$$

$$x_{2h} = a_2 \cos(\omega_0 t + \beta_2)$$

$$x_2 = \underbrace{\frac{\alpha_2 a_0^2}{6\omega_0^2}\left[\cos(2\omega_0 t + 2\beta_0) - 3\right]}_{x_{2p}} + a_2 \underbrace{\cos(\omega_0 t + \beta_2)}_{x_{2h}}$$

$$x_2(0) = 0, \quad \dot{x}_2(0) = 0$$

$$5 \Rightarrow \quad a_2 = \checkmark, \quad \beta_2 = \checkmark$$

$$x = \varepsilon x_1 + \varepsilon^2 x_2 = \varepsilon a_0 \cos(\omega_0 t + \beta_0) + \varepsilon^2 \left\{ \frac{\alpha_2 a_0^2}{6\omega_0^2}\left[\cos(2\omega_0 t + 2\beta_0) - 3\right] + a_2 \cos(\omega_0 t + \beta_2) \right\}$$

$$= \left[\varepsilon a_0 \cos(\omega_0 t + \beta_0) + \varepsilon^2 a_2 \cos(\omega_0 t + \beta_2) \right] + \varepsilon^2 \left\{ \frac{\alpha_2 a_0^2}{6\omega_0^2}\left[\cos(2\omega_0 t + 2\beta_0)\right] \right\}$$

I

روش دوم: در x_2 فقط جواب خصوصی در نظر گرفته می‌شود

$$x_1 = a\cos(\omega_0 t + \beta)$$

$$x_2 = \frac{\alpha_2 a^2}{6\omega_0^2}\left[\cos(2\omega_0 t + 2\beta) - 3\right]$$

$$x = \varepsilon x_1 + \varepsilon^2 x_2 = \varepsilon a\cos(\omega_0 t + \beta) + \frac{\varepsilon^2 \alpha_2 a^2}{6\omega_0^2}\left[\cos(2\omega_0 t + 2\beta) - 3\right] + O(\varepsilon^3)$$

$$x = \underbrace{\overbrace{\varepsilon a \cos(\omega_0 t + \beta)}^{1}} + \overbrace{\frac{\varepsilon^2 a^2 \alpha_2}{6\omega_0^2}\left[\cos(2\omega_0 t + 2\beta) - 3\right]}^{2} + O(\varepsilon^3)$$

$$\varepsilon a = \varepsilon A_1 + \varepsilon^2 A_2 + \varepsilon^3 A_3 + \cdots, \quad \beta = B_0 + \varepsilon B_1 + \varepsilon^2 B_2 + \cdots$$

$$\varepsilon a \cos(\omega_0 t + \beta) = (\varepsilon A_1 + \varepsilon^2 A_2 + \cdots)\cos(\underbrace{\omega_0 t + B_0} + \underbrace{\varepsilon B_1 + \cdots})$$

$$= (\varepsilon A_1 + \varepsilon^2 A_2 + \cdots)\Big[\cos(\omega_0 t + B_0)\underbrace{\cos(\varepsilon B_1 + \varepsilon^2 B_2 + \cdots)}_{\simeq 1}$$

$$- \sin(\omega_0 t + B_0)\underbrace{\sin(\varepsilon B_1 + \varepsilon^2 B_2 + \cdots)}_{\simeq \varepsilon B_1}\Big]$$

$$= \varepsilon A_1 \cos(\omega_0 t + B_0) + \varepsilon^2\left[A_2 \cos(\omega_0 t + B_0) - A_1 B_1 \sin(\omega_0 t + B_0)\right] + O(\varepsilon^3) \quad 3$$

$$= \varepsilon A_1 \cos(\omega_0 t + B_0) + \varepsilon^2\left[(A_2^2 + A_1^2 B_1^2)^{\frac{1}{2}} \cos(\omega_0 t + \theta_2)\right]$$

W

$$\varepsilon a\cos(\omega_0 t+\beta)=\varepsilon A_1\cos(\omega_0 t+B_0)+\varepsilon^2\Big[\underbrace{(A_2^2+A_1^2B_1^2)^{\frac{1}{2}}}_{q}\cos(\omega_0 t+\theta_2)\Big]$$

که: $\theta_2 = B_0 + tg^{-1}\dfrac{A_1B_1}{A_2}$

برای جواب خصوصی

V

$$\frac{\varepsilon^2 a^2\alpha_2}{6\omega_0^2}\Big[\cos(2\omega_0 t+2\beta)-3\Big]=\frac{\varepsilon^2A_1^2\alpha_2}{6\omega_0^2}\Big[\cos(\underbrace{2\omega_0 t+2B_0}+\underbrace{2\varepsilon B_1+\cdots})-3\Big]$$

$$=\frac{\varepsilon^2A_1^2\alpha_2}{6\omega_0^2}\Big[\cos(2\omega_0 t+2B_0)\underbrace{\cos(2\varepsilon B_1+\cdots)}_{\simeq 1}-\sin(2\omega_0 t+2B_0)\sin(\underbrace{2\varepsilon B_1+\cdots}_{2\varepsilon B_1})-3\Big]$$

$$=\frac{\varepsilon^2A_1^2\alpha_2}{6\omega_0^2}\Big[\cos(2\omega_0 t+2B_0)-3\Big]+O(\varepsilon^3)$$

$$x = \varepsilon a \cos(\omega_0 t + \beta) + \frac{\varepsilon^2 a^2 \alpha_2}{6\omega_0^2}\left[\cos(2\omega_0 t + 2\beta) - 3\right] + O(\varepsilon^3)$$

$$= \varepsilon A_1 \cos(\omega_0 t + B_0) + \varepsilon^2 q \cos(\omega_0 t + \theta_2) + \frac{\varepsilon^2 A_1^2 \alpha_2}{6\omega_0^2}\left[\cos(2\omega_0 t + 2B_0) - 3\right] + O(\varepsilon^3) \qquad \text{II}$$

اعمال شرایط اولیه $x(0) = \varepsilon a_0 \cos\beta_0 = \varepsilon A_1 \cos B_0 + O(\varepsilon^2) \Rightarrow \begin{cases} A_1 = a_0 \\ B_0 = \beta_0 \end{cases}$

حال در حل روش اول I با انتخاب

$$a_2 = q \quad \text{و} \quad \beta_2 = \theta_2 = B_0 + \tan^{-1}\frac{A_1 B_1}{A_2}$$

I و II مساوی خواهند شد و هر دو روش به یک جواب می‌رسند.

III

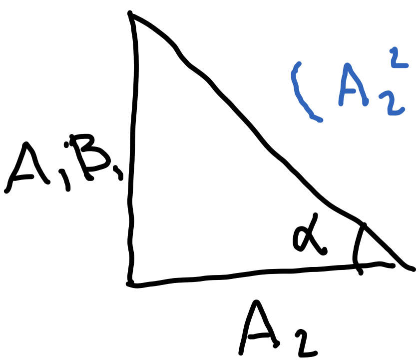
$$\theta_2 = B_0 + tg^{-1}\frac{A_1B_1}{A_2}$$

$$q = \left((A_1B_1)^2 + A_2^2\right)^{\frac{1}{2}}$$

$$\Rightarrow q\cos(\omega_0 t + \theta_2) =$$

$$= q\cos\left(\omega_0 t + B_0 + tg^{-1}\frac{A_1B_1}{A_2}\right) =$$

$$q\left[\cos(\omega_0 t + B_0)\cos\left[\underbrace{tg^{-1}\frac{A_1B_1}{A_2}}_{\alpha}\right] - \sin(\omega_0 t + B_0)\sin\left[\underbrace{tg^{-1}\frac{A_1B_1}{A_2}}_{\alpha}\right]\right]$$



$$\left(A_2^2 + A_1^2B_1^2\right)^{\frac{1}{2}} = q$$

$$tg\,\alpha = \frac{A_1B_1}{A_2} \Rightarrow \alpha = tg^{-1}\frac{A_1B_1}{A_2}$$

$$\cos\alpha = \frac{A_2}{q}$$

$$\sin\alpha = A_1B_1/q$$

$$= q\cos(\omega_0 t + B_0)\cos\alpha - q\sin(\omega_0 t + B_0)\sin\alpha$$

$$= A_2\cos(\omega_0 t + B_0) - A_1B_1\sin(\omega_0 t + B_0)$$