ارتعاشات غیر خطی

Perturbation theory, Duffing Equation

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Duffing eq: $\ddot{u} + u + \varepsilon u^3 = 0$, $\varepsilon \ll 1$, cubic nonlinearity

straight forward expansion, $u = u_0 + \varepsilon u_1 + O(\varepsilon^2)$

$\varepsilon^0: \ \ddot{u}_0 + u_0 = 0 \longrightarrow u_0 = a\cos(t+\beta)$ I

$\varepsilon^1: \ \ddot{u}_1 + u_1 = -u_0^3 \longrightarrow \ddot{u}_1 + u_1 = -a^3\cos^3(t+\beta)$

$\cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$

$\Rightarrow \ \ddot{u}_1 + u_1 = -\frac{a^3}{4}\cos 3\theta - \frac{3a^3}{4}\cos\theta$

$$\ddot{u}_1 + u_1 = \underbrace{-\frac{a^3}{4}\cos 3\theta}_{1} \; \underbrace{-\frac{3a^3}{4}\cos\theta}_{2}, \qquad \theta = t+\beta$$

(1) $u_1 = A\cos 3\theta + B\sin 3\theta \longrightarrow \dot{u}_1 = -3A\sin(3t+3\beta) + 3B\cos(3t+3\beta)$

$\rightarrow \ddot{u}_1 = -9A\cos(3t+3\beta) - 9B\sin(3t+3\beta)$

$-8A\cos(3t+3\beta) - 8B\sin(3t+3\beta) = -\frac{a^3}{4}\cos(3t+3\beta)$

$\Rightarrow \quad B = 0, \quad -8A = \frac{-a^3}{4} \Rightarrow A = \frac{a^3}{32}$

(2) $u_1 = At\sin(t+\beta) \longrightarrow \dot{u}_1 = \checkmark, \quad \ddot{u}_1 = \checkmark \Rightarrow A = \frac{-3a^3}{8}$

جواب خصوصی: $u_1 = \frac{1}{32}a^3\cos(3t+3\beta) - \frac{3}{8}ta^3\sin(t+\beta)$ II

$\Rightarrow \boxed{u = \underbrace{u_0}_{\mathrm{I}} + \varepsilon \overbrace{u_1}^{\mathrm{II}}}$ SFE

حل معادله دافینگ بارو‌ش Multiple scale

$$\ddot{u} + u + \varepsilon u^3 = 0 , \qquad u(t,\varepsilon) = u_0(T_0, T_1) + \varepsilon u_1(T_0, T_1) + \cdots$$

$$\varepsilon^0 : \quad D_0^2 u_0 + u_0 = 0 \quad \Rightarrow \quad u_0 = a(T_1)\cos(T_0 + \beta(T_1))$$

$$\varepsilon^1 : \quad D_0^2 u_1 + u_1 = -2 D_1 D_0 u_0 - u_0^3$$

$$a' = \frac{\partial a(T_1)}{\partial T_1}$$

$$-2D_1(D_0 u_0) = -2D_1\left(-a(T_1)\sin(T_0 + \beta(T_1))\right) ,$$

$$= 2a'\sin(T_0 + \beta) + 2a\beta'\cos(T_0 + \beta)$$

$$-u_0^3 = -a^3 \cos^3(T_0 + \beta) = -\frac{a^3}{4}\cos(3T_0 + 3\beta) - \frac{3}{4}a^3\cos(T_0 + \beta)$$

$$D_0^2 u_1 + u_1 = \underline{2a'\sin(T_0+\beta)} + \underline{2a\beta'\cos(T_0+\beta)} + -\frac{a^3}{4}\cos(3T_0+3\beta) - \underline{\frac{3}{4}a^3\cos(T_0+\beta)}$$

Eliminating secular term: $\underline{a'=0} \Rightarrow \boxed{a=a_0=}$ ثابت

$$\Rightarrow \underline{2a\beta' - \frac{3a^3}{4}} = 0 \xRightarrow{a\neq 0} \beta' = \frac{3a_0^2}{8} \Rightarrow \boxed{\beta = \frac{3a_0^2}{8}T_1+\beta_0}$$

$$D_0^2 u_1 + u_1 = -\frac{a^3}{4}\cos(3T_0+3\beta)$$ جواب خصوصی $$\Rightarrow u_1 = A\cos(3T_0+3\beta)$$

$$u_1 = \frac{a^3}{32}\cos(3T_0+3\beta)$$

$$\Rightarrow u = u_0+\varepsilon u_1 = a\cos(T_0+\beta) + \frac{\varepsilon a^3}{32}\cos(3T_0+3\beta)$$

$$\Rightarrow u = a_0 \cos\left(T_0 + \frac{3a_0^2 T_1}{8} + \beta_0\right) + \frac{\varepsilon a_0^3}{32}\cos\left(3T_0 + \frac{9a_0^2 T_1}{8} + 3\beta_0\right) + O(\varepsilon^2)$$

$$\Rightarrow \boxed{u = a_0 \cos\left(t + \frac{3a_0 \varepsilon t}{8} + \beta_0\right) + \frac{\varepsilon a_0^3}{32}\cos\left(3t + \frac{9a_0 \varepsilon t}{8} + 3\beta_0\right)}$$

MS

تمرین حل معادله دافینگ با سه روش

1- SFE

2- MS

3- Numerical ← ode45

MATLAB جواب سه روش را مقایسه کنید ←

$u(0)=1$ ، $\dot{u}(0)=0$ و $\varepsilon=0.1$

محاسبه $\cos^n(\theta)$

$$\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$$

$$\cos^2\theta = \frac{1}{4}\left(e^{2i\theta} + e^{-2i\theta} + 2\right) = \frac{1}{2}\left[\frac{e^{2i\theta} + e^{-2i\theta}}{2}\right] + \frac{1}{2}$$

$$= \frac{1}{2}\left(\cos 2\theta + 1\right)$$

$$\cos^3\theta = \frac{1}{8}\left(e^{3i\theta} + 3e^{i\theta} + 3e^{-i\theta} + e^{-3i\theta}\right) = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$$

$$(a+b)^3 = a^3 + 3a^2 b + 3ab^2 + b^3$$

$$(a+b)^1 = a+b$$

$$(a+b)^2 = a^2+2ab+b^2 \leftarrow 1 \quad 2 \quad 1$$

$$(a+b)^3 = a^3+3a^2b+3ab^2+b^3 \leftarrow 1 \quad 3 \quad 3 \quad 1 \rightarrow 3$$

$$\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1
\end{array}
\qquad
\begin{array}{l}
\rightarrow 3 \\
\rightarrow 4 \\
\rightarrow 5 \\
\rightarrow 6 \\
\rightarrow 7
\end{array}$$

$$(a+b)^5 = a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$$

$e^{i\theta}$ $\quad$ $e^{-i\theta}$

$$\sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i} \rightarrow (a-b)^n$$

$(a-b)^2$

$(a -$

$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad -2 \quad 1 \\
1 \quad -3 \quad 3 \quad -1 \\
1 \quad -4 \quad 6 \quad -4 \quad 1 \\
1 \quad -5 \quad 10 \quad -10 \quad 5 \quad -1 \\
1 \quad -6 \quad 15 \quad -20 \quad 15 \quad -6 \quad 1 \\
1 \quad -7 \quad 21 \quad -35 \quad 35 \quad -21 \quad 7 \quad -1 \\
\vdots
\end{array}
$$

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$

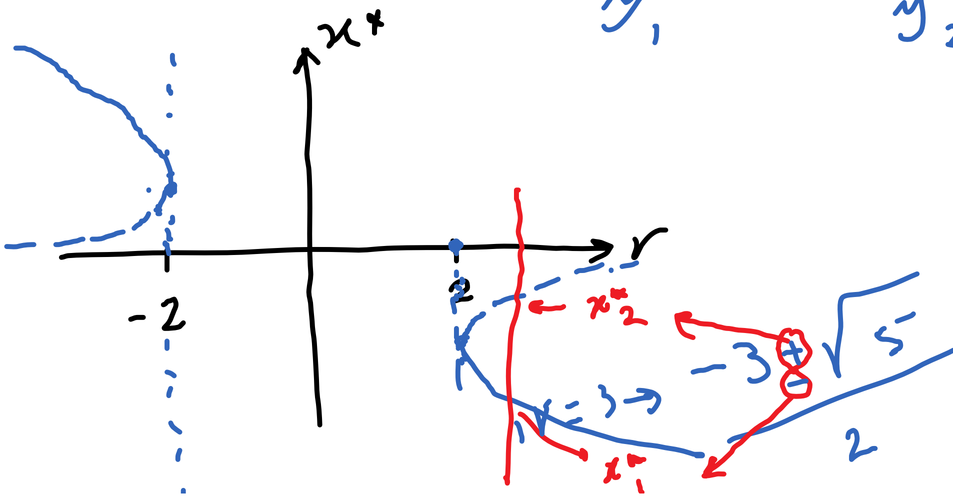
3.1.1

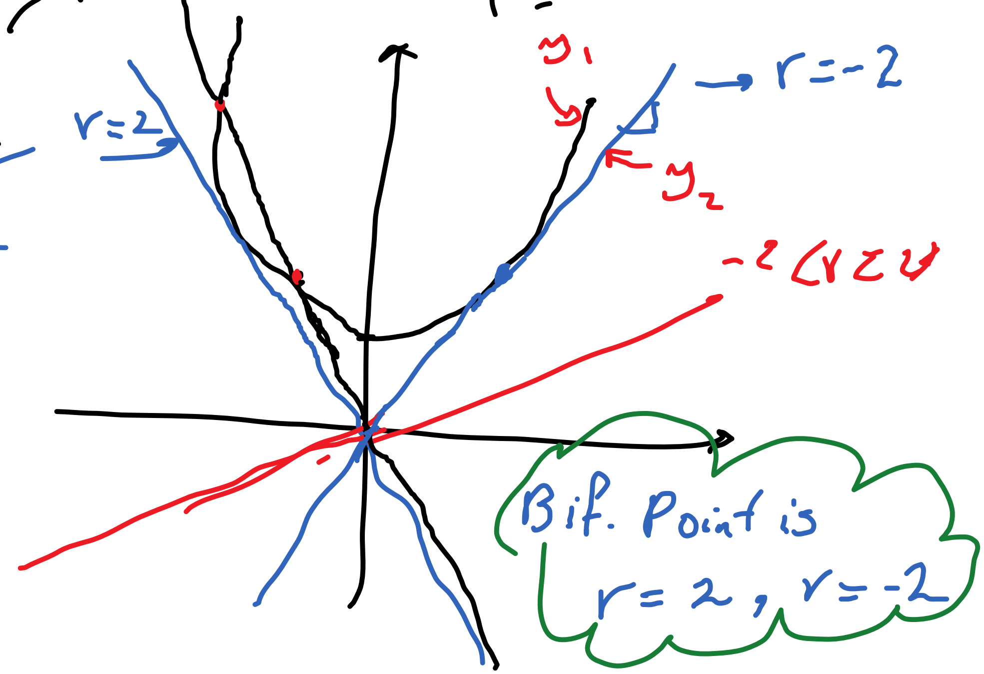
$$\dot{x} = 1 + rx + x^2 = f(x) \qquad \dot{x} = 0 \Rightarrow x^2 + rx + 1 = 0$$

$$x^*_{1,2} = \frac{-r \pm \sqrt{r^2 - 4}}{2}, \qquad -2 < r < 2$$ نقطه ثابت نداریم

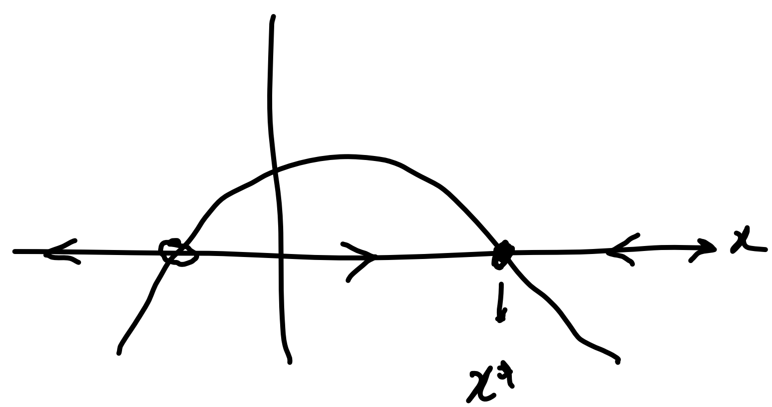
$r > +2$, $r < -2$ دو نقطه ثابت داریم

$$1 + x^2 + rx = 0 \Rightarrow \underbrace{1 + x^2}_{y_1} = \underbrace{-rx}_{y_2}$$





$\dot{x} = f(x)$



$f'(x^*) < 0 \rightarrow x^*$ پایدار

$f'(x^*) > 0 \longrightarrow x^*$ ناپایدار

$f(x) = x^2 + rx + 1 \longrightarrow f'(x) = 2x + r \Rightarrow f'(x^*) = 2x^* + r$

$x^*_{1,2} = \dfrac{-r \pm \sqrt{r^2 - 4}}{2} \Rightarrow f'(x^*) = -r \pm \sqrt{r^2 - 4} + r$

x^*_1 (1) $\rightarrow f'(x^*_1) = + \sqrt{r^2 - 4} > 0 \longrightarrow$ ناپایدار

$x^*_2 \longrightarrow f'(x^*_2) = - \sqrt{r^2 - 4} < 0 \longrightarrow$ پایدار