ارتعاشات غیر خطی

Perturbation Theory, Physical Observation

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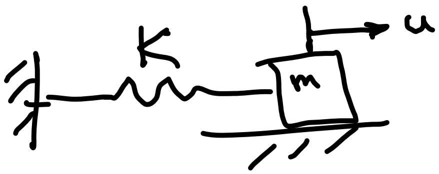
Physical observation



$$m\ddot{u} + ku = 0 \xrightarrow[m=1]{k=1} \ddot{u} + \underbrace{u}_{F(u)} = 0 \Rightarrow \ddot{u}\,du + u\,du = 0$$

$$\Rightarrow \dot{u}\,d\dot{u} + u\,du = 0 \Rightarrow \int \dot{u}\,d\dot{u} + \int u\,du = 0$$

$$\Rightarrow \frac{1}{2}\dot{u}^2 + \frac{1}{2}u^2 = H =$$ مقدار انرژی کل

$K + U = H$

انرژی جنبشی ← $\frac{1}{2}\dot{u}^2$

انرژی پتانسیل $= G(u) = \int F(u)\,du$

$$\frac{1}{2}\dot{u}^2 + G(u) = H$$

فرم کلی مسئله

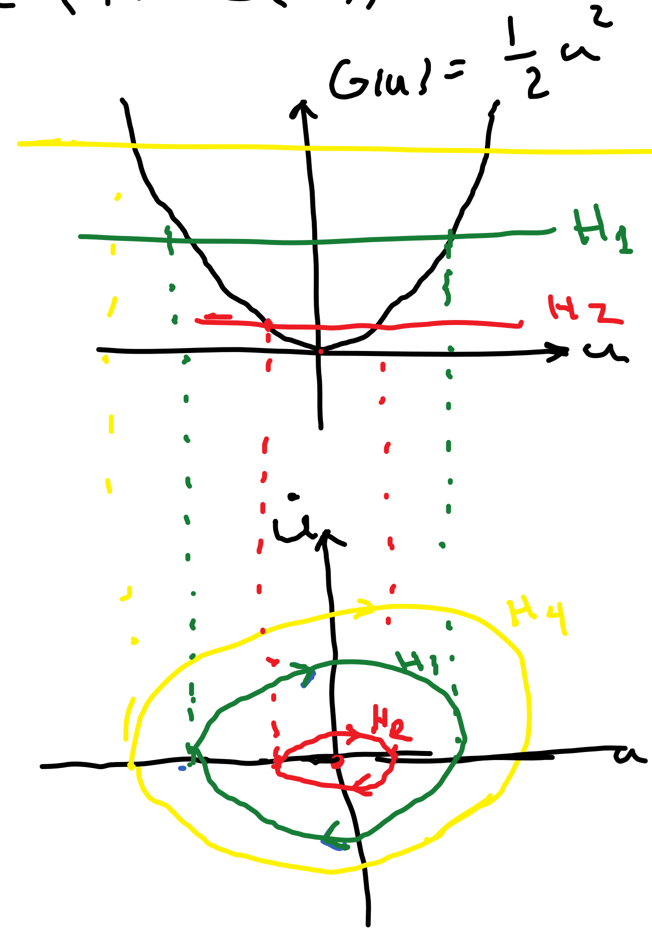
$$\ddot{u} + F(u) = 0 \Rightarrow \ddot{u}\,du + F(u)\,du = 0 \Rightarrow \dot{u}\,d\dot{u} + F(u)\,du = 0$$

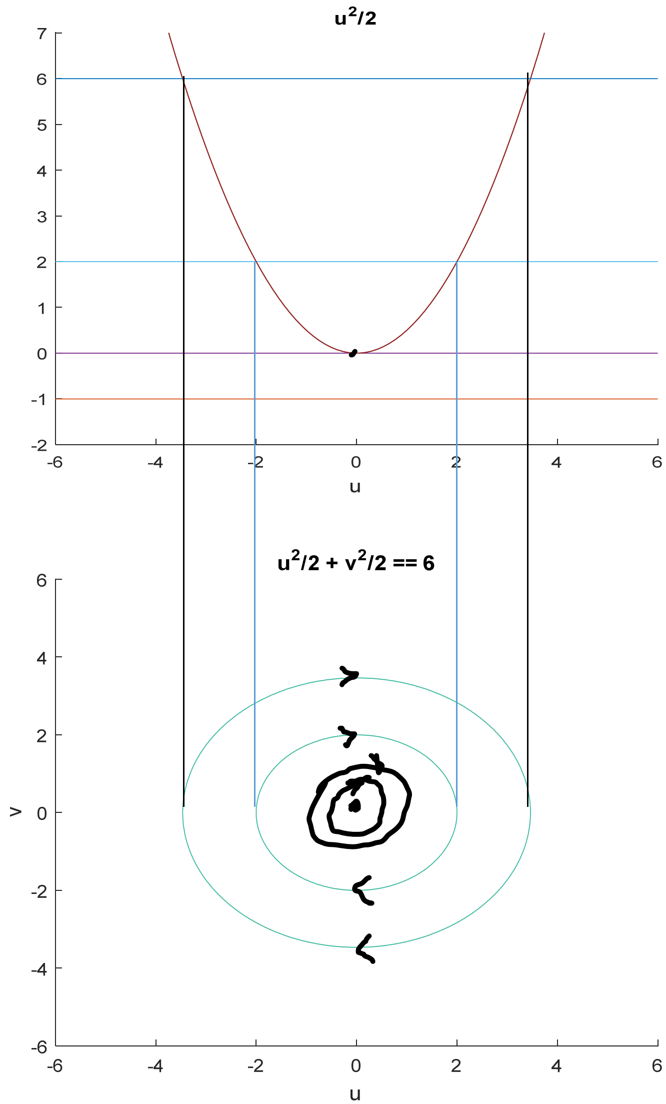
$$\Rightarrow \int \dot{u}\,d\dot{u} + \int F(u)\,du = H \Rightarrow \frac{1}{2}\dot{u}^2 + G(u) = H, \quad G(u) = \int F(u)\,du$$

$$\Rightarrow \frac{1}{2}\dot{u}^2 + G(u) = H \Rightarrow \dot{u}^2 = 2\left(H - G(u)\right)$$

$$\Rightarrow \dot{u} = \pm\sqrt{2}\left(H - G(u)\right)^{\frac{1}{2}}$$

فنر خطی : $\frac{1}{2}\dot{u}^2 + \frac{1}{2}u^2 = H$




$u^2/2$
7
6
5
4
3
2
1
0
-1
-2
-6
-4
-2
0
2
4
6
u
$u^2/2 + v^2/2 == 6$
6
4
2
v
0
-2
-4
-6
-6
-4
-2
0
2
4
6
u

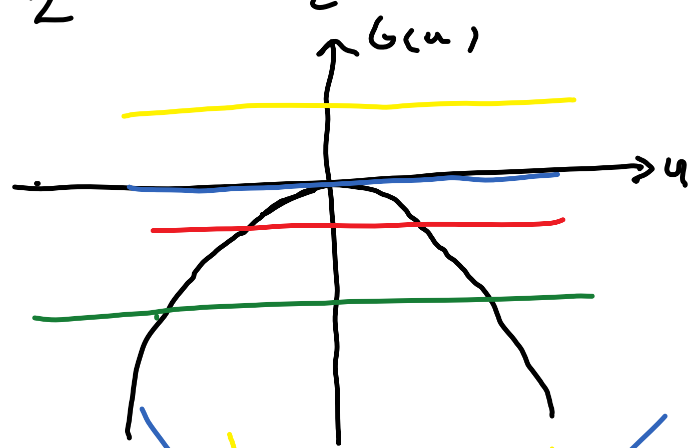
$$\ddot{u} - u = 0 \Rightarrow \frac{1}{2}\dot{u}^2 - \frac{1}{2}u^2 = H \quad , \quad G(u) = -\frac{1}{2}u^2$$

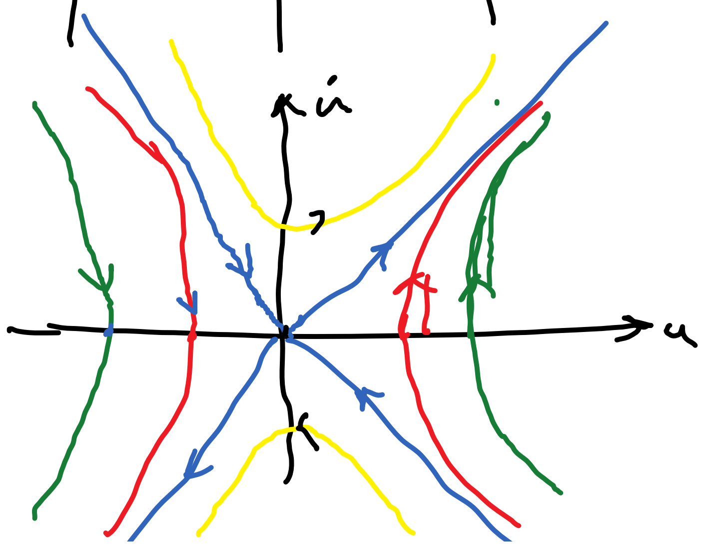
$$u = u_1$$

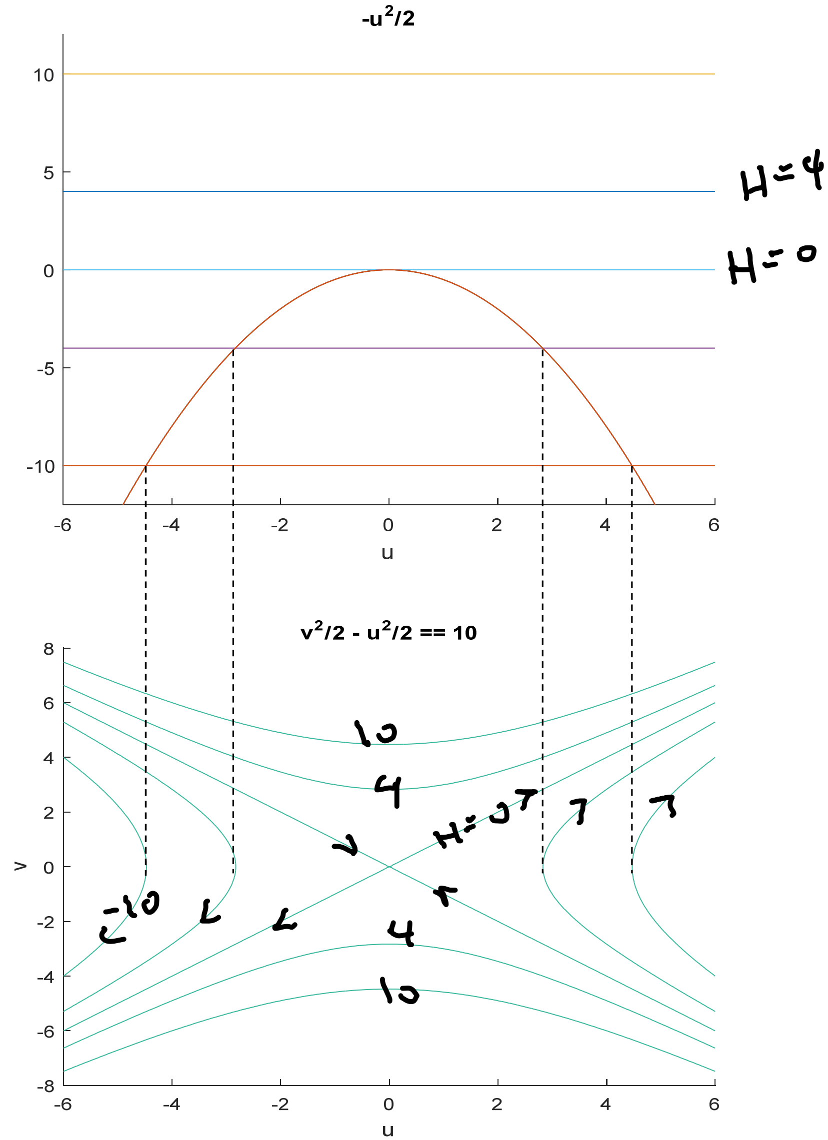
$$\dot{u} = u_2$$

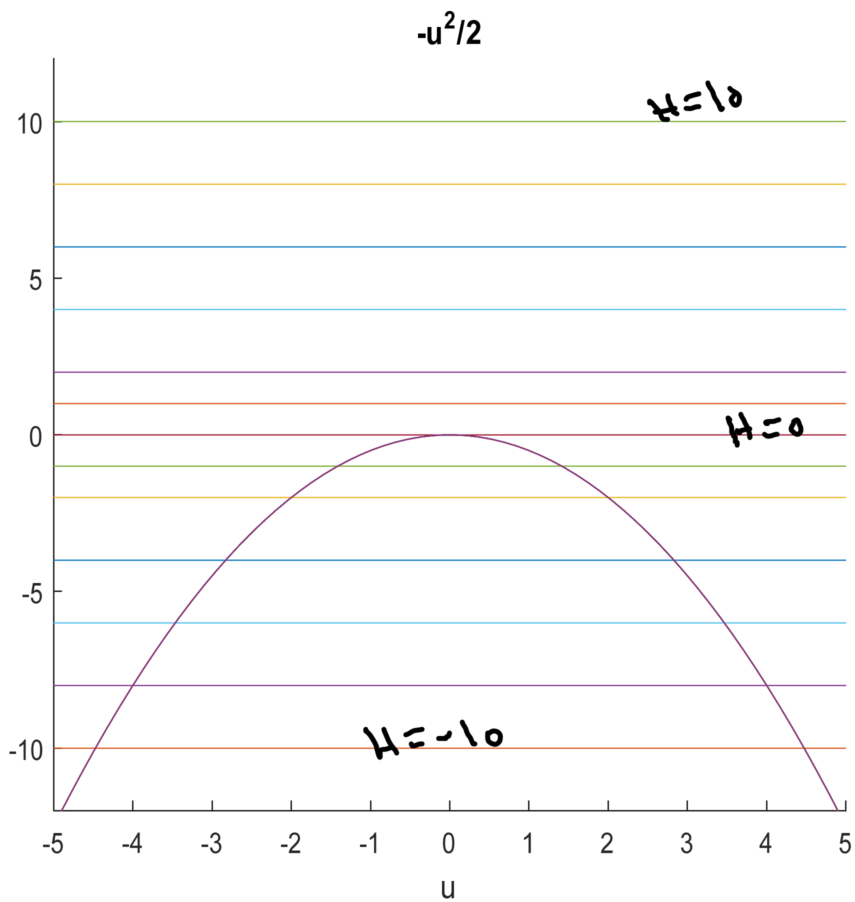
$$\begin{cases} \dot{u}_1 = u_2 \\ \dot{u}_2 = u_1 \end{cases}$$

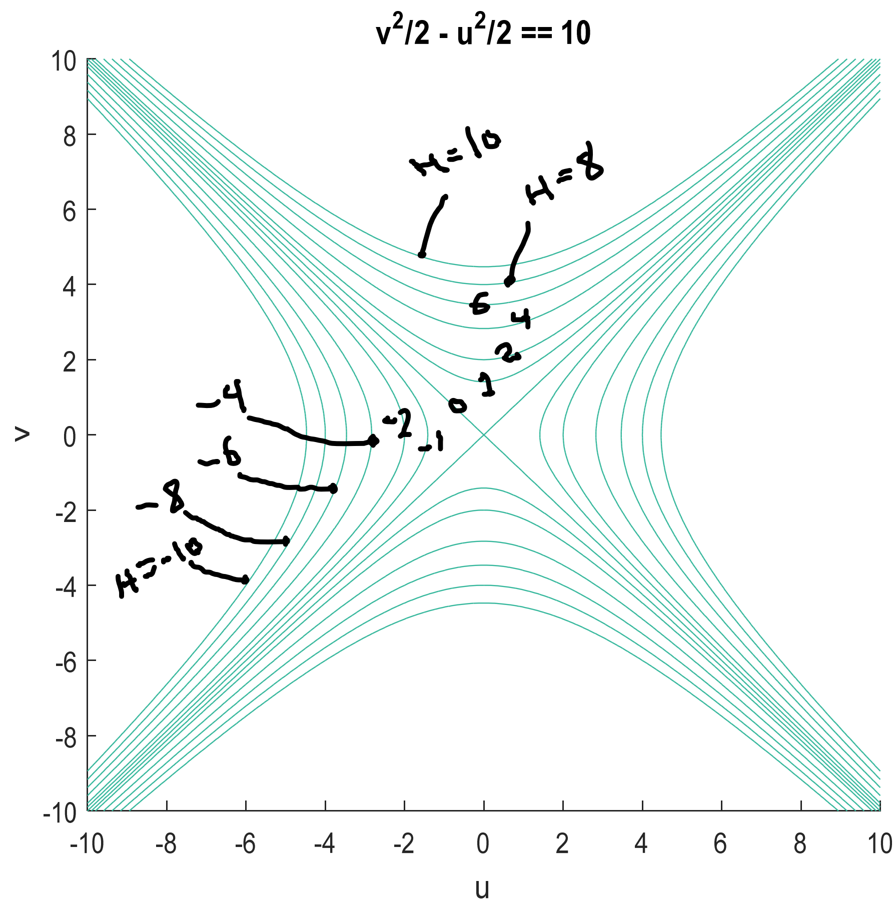
$$\lambda_{1,2} = \pm 1$$






-u²/2
u
v²/2 - u²/2 == 10
v
u
H=۴
H=۰
۱۰
۴


-u^2/2
H=10
H=0
H=-10
u


v^2/2 - u^2/2 == 10
H=10
H=8
6
4
2
1
0
-1
-2
-4
-6
-8
H=-10
v
u

Duffing Eq: $\ddot{u} + u + \varepsilon u^3 = 0 \Rightarrow \ddot{u}\,du + (u + \varepsilon u^3)\,du = 0$

$$\Rightarrow \frac{1}{2}\dot{u}^2 + G(u) = H, \quad G(u) = \int (u + \varepsilon u^3)\,du = \frac{1}{2}u^2 + \frac{1}{4}\varepsilon u^4$$

$$\begin{cases} \dot{u} = v \\ \dot{v} = -u - \varepsilon u^3 \end{cases} \xrightarrow{\text{Fixed Point}} v = 0$$

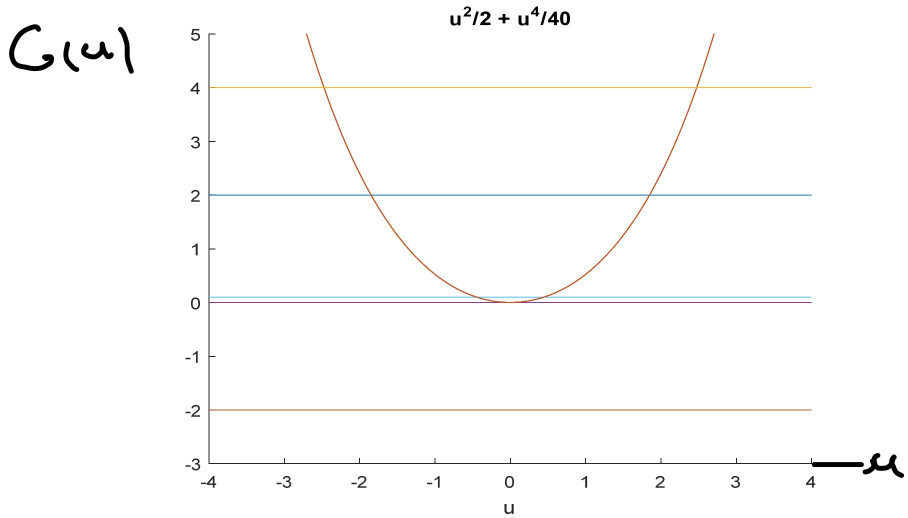
$$-u - \varepsilon u^3 = 0 \Rightarrow -u(1 + \varepsilon u^2) = 0 \Rightarrow u = 0$$

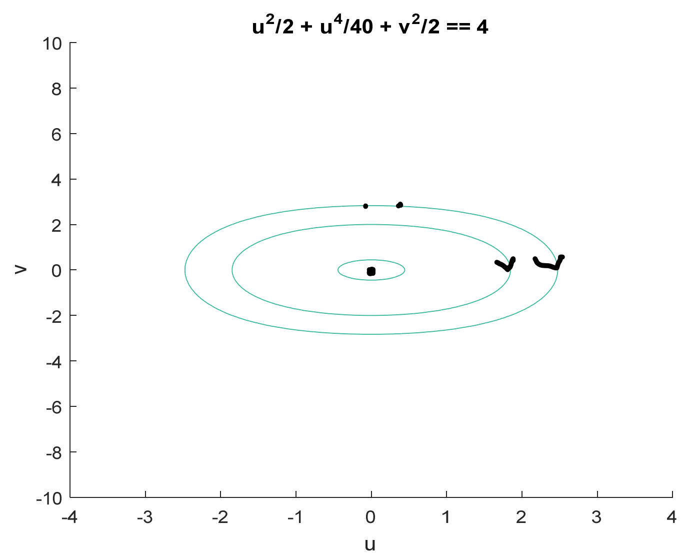
نقطه تعادل $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$A = \begin{bmatrix} 0 & 1 \\ -1 - 3\varepsilon u^2 & 0 \end{bmatrix}\Bigg|_{*} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\det(A - \lambda I) = \det\begin{bmatrix} -\lambda & 1 \\ -1 & -\lambda \end{bmatrix} = \lambda^2 + 1 = 0 \longrightarrow \lambda_{1,2} = \pm i \longrightarrow$$

نقطه تعادل center – برای سیستم‌های پایستار اگه خطی‌سازی نقطه تعادل را center بدست آورد، نقطه تعادل حتماً center است.


G(u)
$u^2/2 + u^4/40$
5
4
3
2
1
0
-1
-2
-3
-4
-3
-2
-1
0
1
2
3
4
u
u


$u^2/2 + u^4/40 + v^2/2 == 4$
10
8
6
4
2
0
-2
-4
-6
-8
-10
v
-4
-3
-2
-1
0
1
2
3
4
u

$$\ddot{u} + u - \varepsilon u^3 = 0 \qquad \varepsilon \ll 1$$

$$\Rightarrow \frac{1}{2}\dot{u}^2 + G(u) = H, \qquad G(u) = \frac{1}{2}u^2 - \frac{1}{4}\varepsilon u^4$$

Fixed po.. $v = 0$

$$\begin{cases} \dot{u} = v \\ \dot{v} = -u + \varepsilon u^3 \end{cases} \Rightarrow -u(1-\varepsilon u^2) = 0 \rightarrow \begin{cases} u = 0 \\ u = \pm\sqrt{\frac{1}{\varepsilon}} \end{cases} \Rightarrow \begin{cases} ① \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ ② \begin{bmatrix} \frac{1}{\sqrt{\varepsilon}} \\ 0 \end{bmatrix} \\ ③ \begin{bmatrix} -\frac{1}{\sqrt{\varepsilon}} \\ 0 \end{bmatrix} \end{cases}$$

$$A = \begin{bmatrix} 0 & 1 \\ -1+3\varepsilon u^2 & 0 \end{bmatrix} \Rightarrow A_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \rightarrow \lambda_{1,2} = \pm i \rightarrow \text{Center}$$

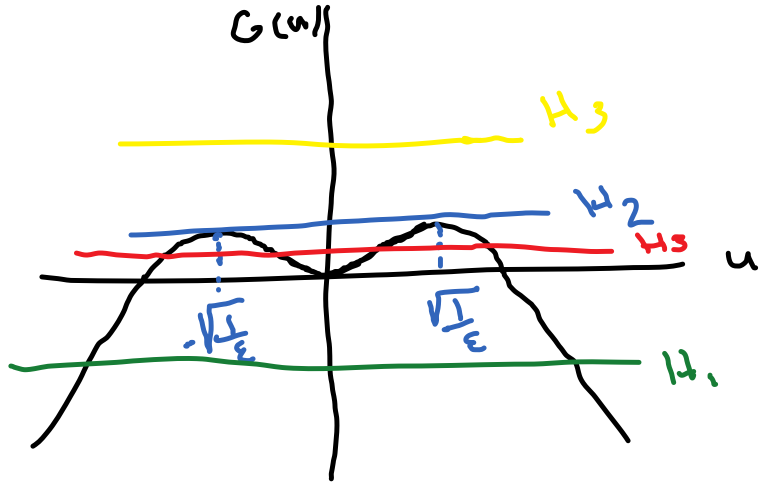
$$A_{2,3} = \begin{bmatrix} 0 & 1 \\ -1+\frac{3\varepsilon}{\varepsilon} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix}$$

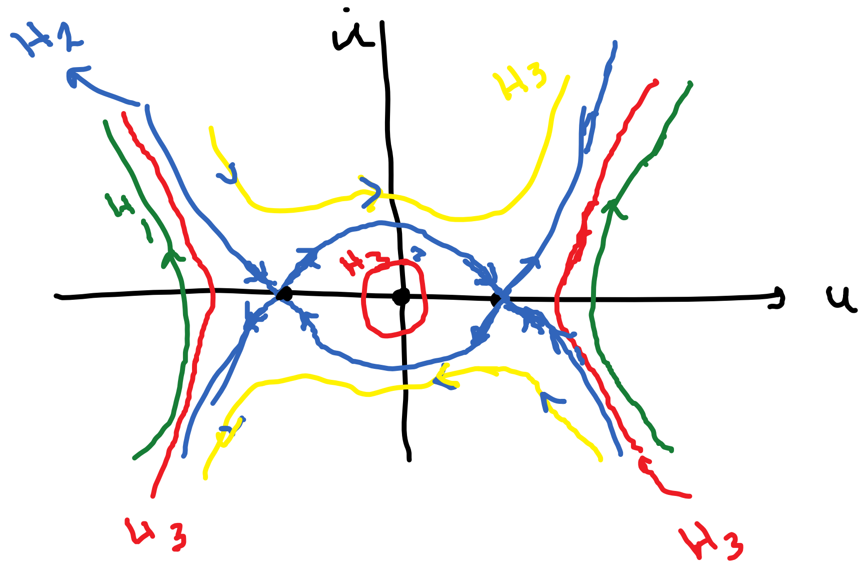
$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ 2 & -\lambda \end{vmatrix} = \lambda^2 - 2 = 0 \rightarrow \begin{cases} \lambda_1 = \sqrt{2} \\ \lambda_2 = -\sqrt{2} \end{cases} \rightarrow \text{saddle}$$

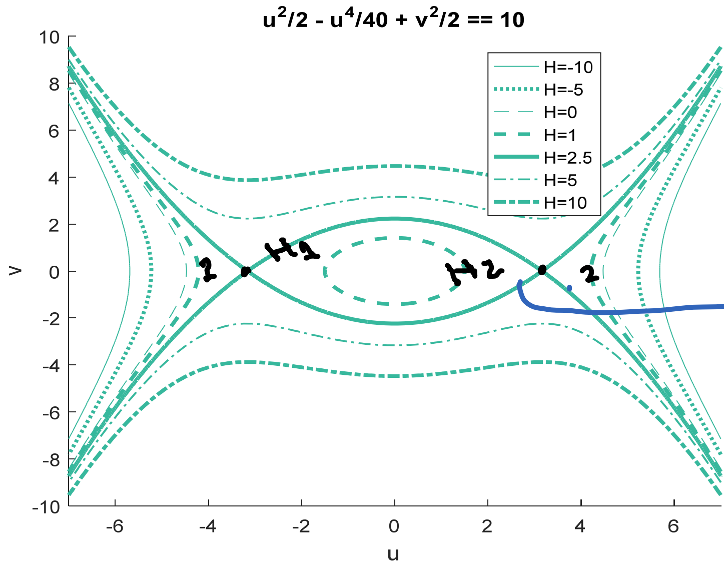
$$G(u) = \frac{1}{2}u^2 - \frac{1}{4}\varepsilon u^4$$

Min $G(u) \longrightarrow$ center

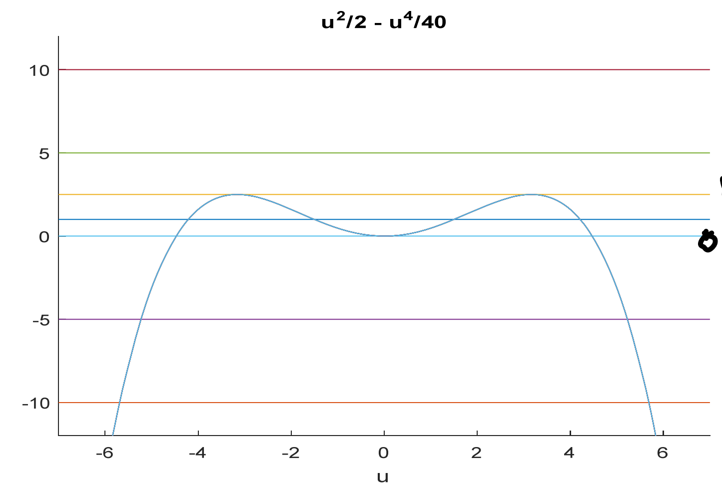
Max $G(u) \longrightarrow$ saddle



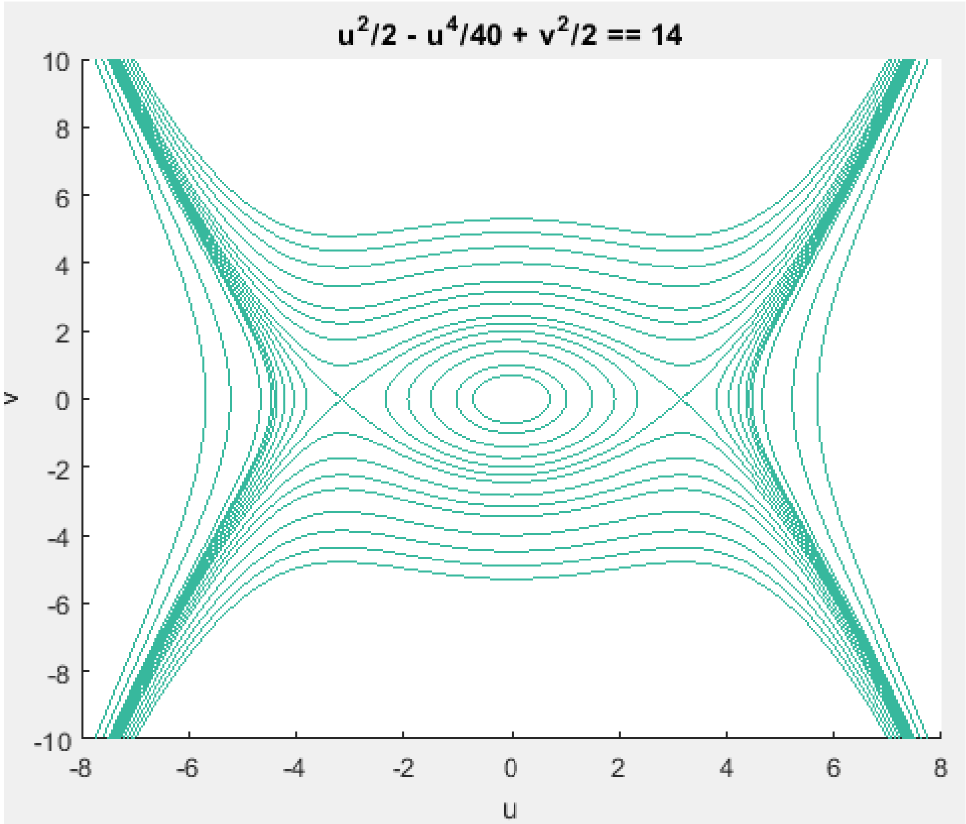


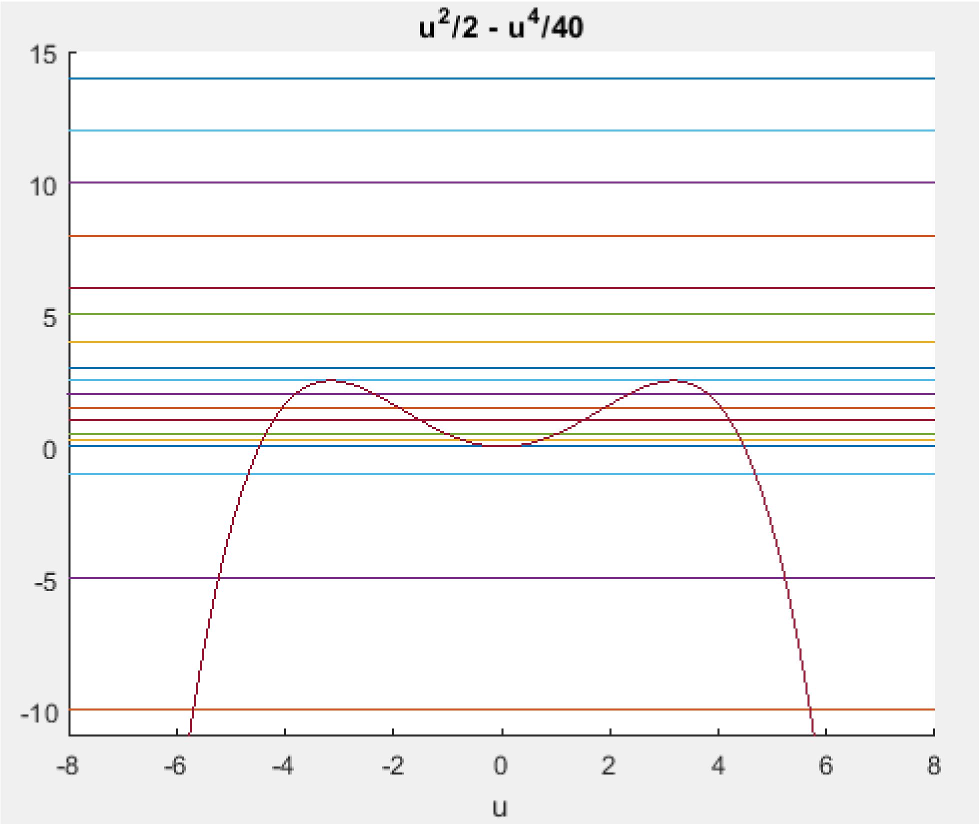


Hetero cilinic orbit

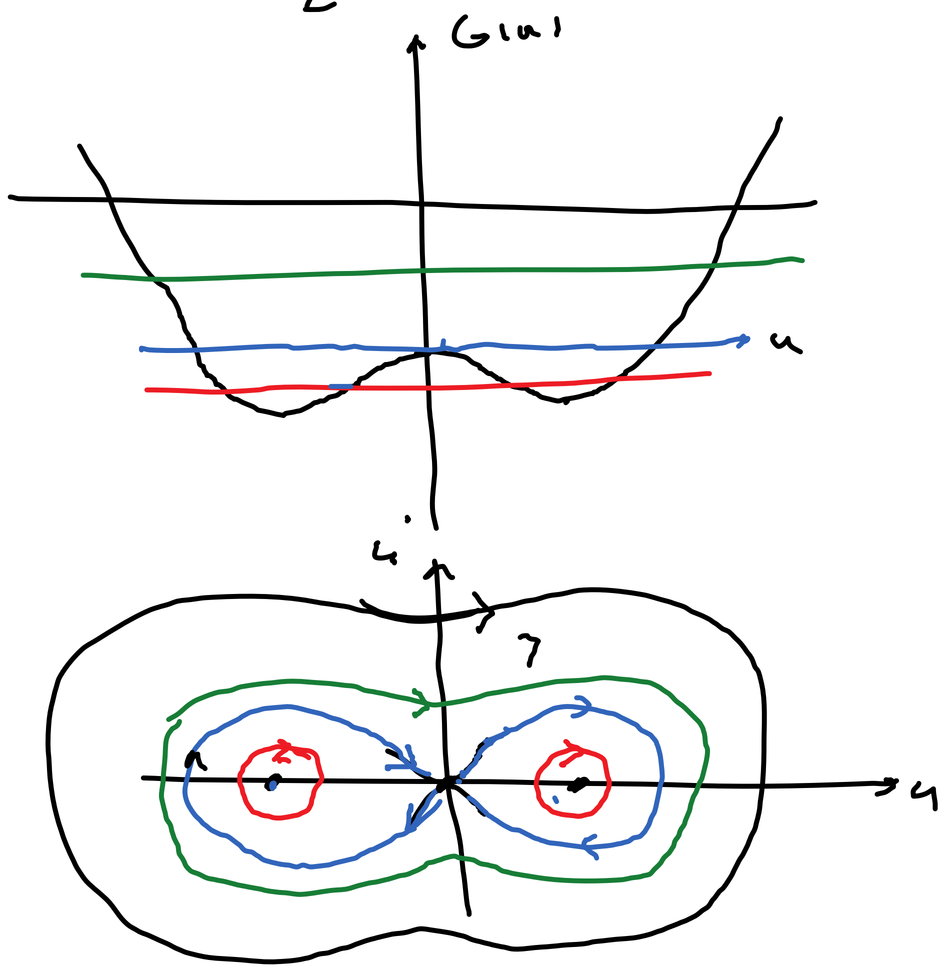


H_1
H_2
0

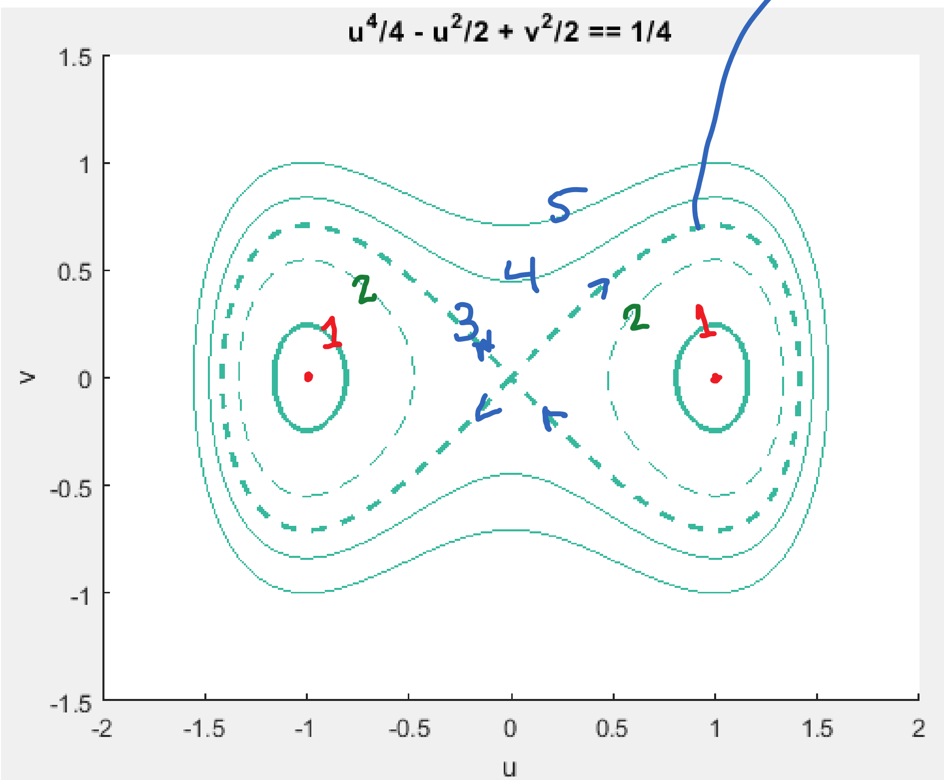

$u^2/2 - u^4/40 + v^2/2 == 14$
10
8
6
4
2
0
-2
-4
-6
-8
-10
v
-8
-6
-4
-2
0
2
4
6
8
u

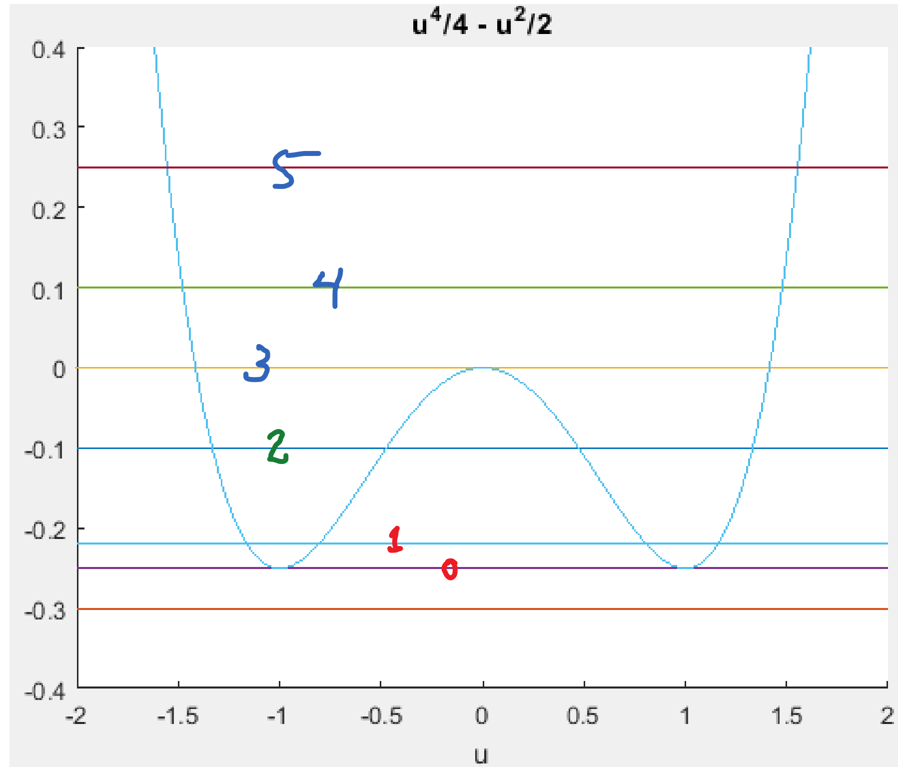

$u^2/2 - u^4/40$
15
10
5
0
-5
-10
-8
-6
-4
-2
0
2
4
6
8
u

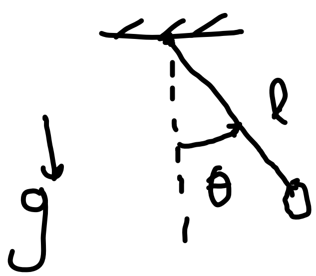
$$\ddot{u} + u^3 - u = 0 \quad \Rightarrow \quad \frac{1}{2}\dot{u}^2 + G(u) = H, \quad G(u) = \frac{1}{4}u^4 - \frac{1}{2}u^2$$



Homoclinic orbit

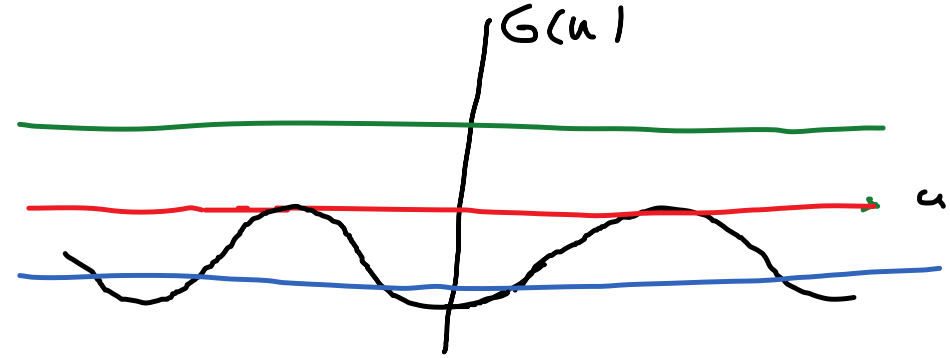






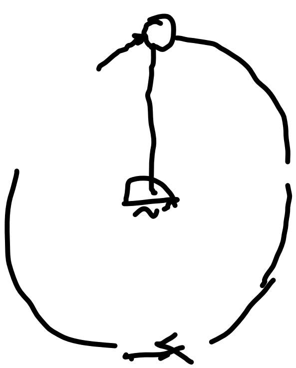
$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0 \Rightarrow \frac{1}{2}\dot{\theta}^2 + G(u) = H$$

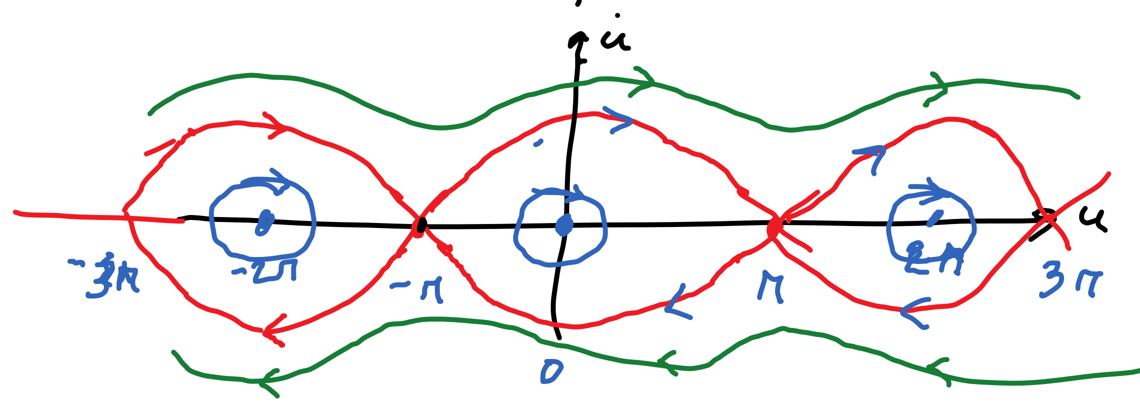
$$G(u) = \int \frac{g}{\ell}\sin\theta = -\frac{g}{\ell}\cos\theta$$

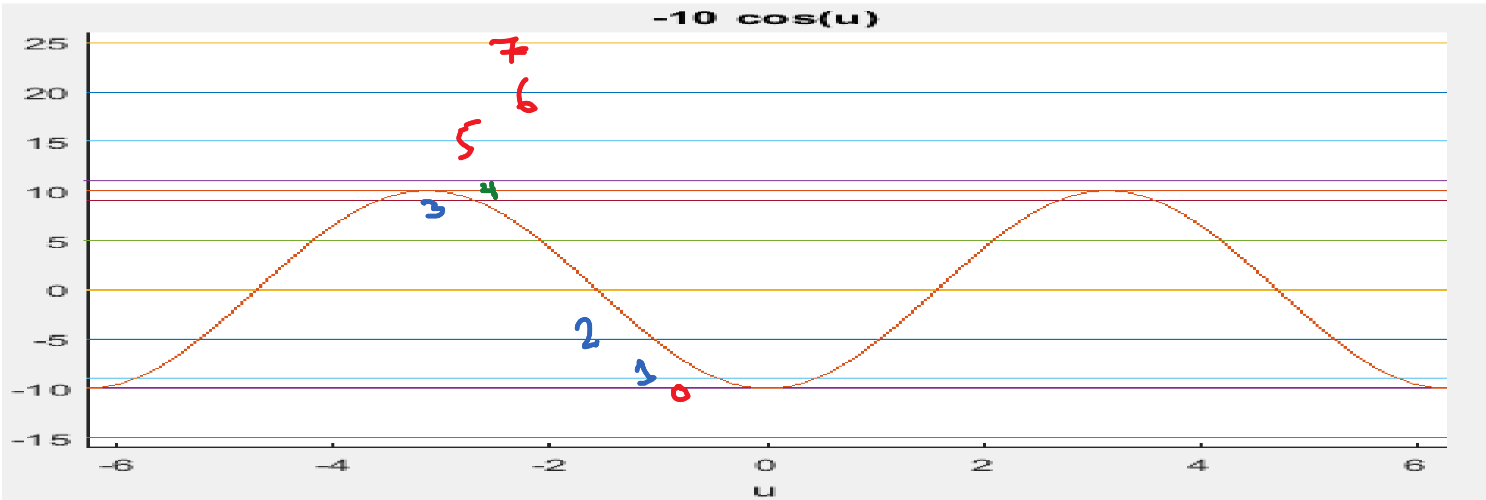


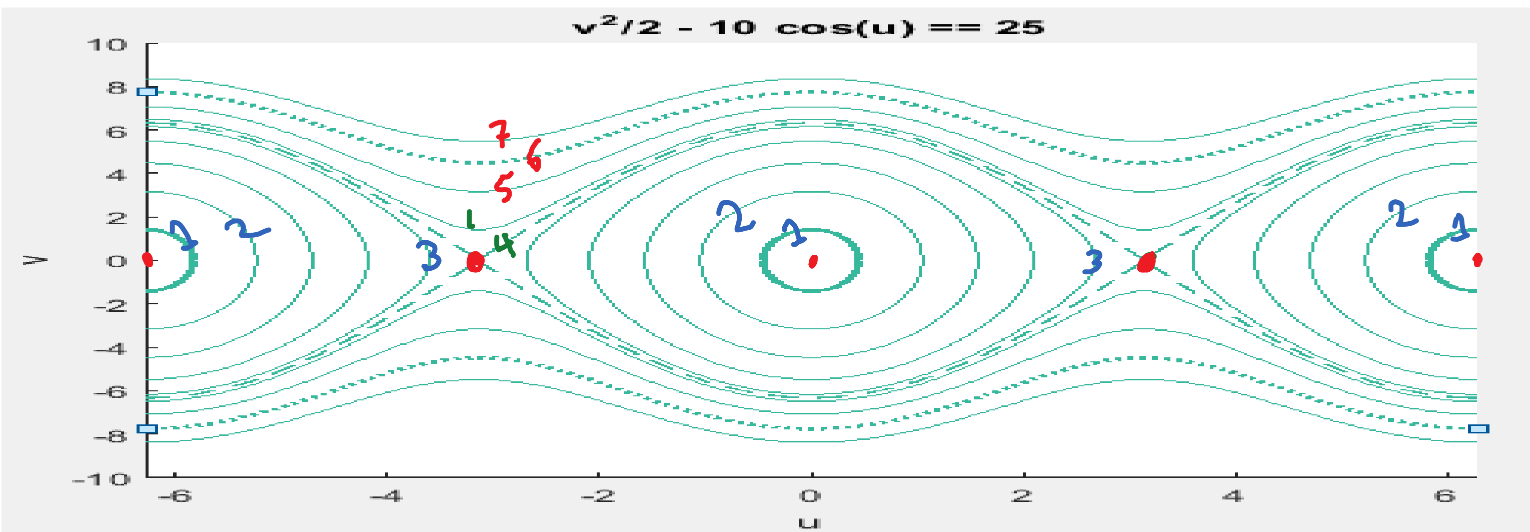
خطی سازی $\rightarrow \begin{bmatrix} n\pi \\ 0 \end{bmatrix} \rightarrow$ n زوج $\rightarrow$ center

n فرد $\rightarrow$ saddle






-10 cos(u)
u
7
6
5
4
3
2
1
0


v²/2 - 10 cos(u) == 25
v
u
7
6
5
4
3
2
1