ارتعاشات غیر خطی

Perturbation Theory
Quadratic Nonlinearity

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Free oscillation of system with quadratic nonlinearities

$$\ddot{u} + u + \varepsilon u^2 = 0$$

اگر غیرخطی زوج باشد روشهای انتگرالی با بسط مرتبه اول (first order exp.) جواب نمی دهد.

برای حل این مسئله از Generalized Averaging به جای Averaging استفاده می کنیم.

↳ Nayfeh ; Perturbation technique

در این روش MS، کافی است از بسط مرتبه اول استفاده کنیم. ✓

$$f = -u^2$$

چرا سطا مرتبه اول جواب کافی نیست،

$$\begin{cases} a' = \dfrac{-1}{2\pi\omega_0} \displaystyle\int_0^{2\pi} f(a\cos\varphi, -a\omega_0 \sin\varphi) \sin\varphi \, d\varphi \\ a\beta' = \dfrac{-1}{2\pi\omega_0} \displaystyle\int_0^{2\pi} f(a\cos\varphi, -a\omega_0 \sin\varphi) \cos\varphi \, d\varphi \end{cases}$$

صفر

$$\Rightarrow \begin{cases} a' = \dfrac{-1}{2\pi} \displaystyle\int_0^{2\pi} -a^2 \cos^2\varphi \sin\varphi \, d\varphi = 0 \\ a\beta' = \dfrac{-1}{2\pi} \displaystyle\int_0^{2\pi} -a^2 \cos^2\varphi \cos\varphi \, d\varphi = 0 \end{cases} \Rightarrow u = a_0 \cos(t + \beta_0)$$

$\cos^3\varphi \rightarrow (\cos\varphi, \cos 3\varphi)$

EX. $\ddot{u} + \omega_0^2 u + \varepsilon \alpha u^2 = 0$, $\varepsilon \ll 1$

MS: using second order expansion: $u = u_0 + \varepsilon u_1 + \varepsilon^2 u_2$

(1) ε^0: $D_0^2 u_0 + \omega_0^2 u_0 = 0 \Rightarrow u_0(T_0, T_1, T_2) = A(T_1, T_2) e^{i\omega_0 T_0} + \bar{A} e^{-i\omega_0 T_0}$

(2) ε^1: $D_0^2 u_1 + \omega_0^2 u_1 = -2 D_0 D_1 u_0 - \alpha u_0^2$

(3) ε^2: $D_0^2 u_2 + \omega_0^2 u_2 = -2 D_0 D_1 u_1 - D_1^2 u_0 - 2 D_0 D_2 u_0 - 2\alpha u_0 u_1$

(2): $-2 D_0 D_1 u_0 = -2 D_1 D_0 u_0 = -2 D_1 \left(i\omega_0 A e^{i\omega_0 T_0} + cc \right)$

$= -2 i\omega_0 D_1 A e^{i\omega_0 T_0} + cc$

$-\alpha u_0^2 = -\alpha \left(A^2 e^{2i\omega_0 T_0} + 2A\bar{A} + \bar{A}^2 e^{-2i\omega_0 T_0} \right)$

$$-2D_0D_1u_0-\alpha u^2=-2i\omega_0 D_1 A e^{i\omega_0 T_0}-\alpha\left(A^2e^{2i\omega_0T_0}+2A\bar{A}\right)+CC$$

Eliminating the secular term: $D_1A=0 \Rightarrow \dfrac{\partial A(T_1,T_2)}{\partial T_1}=0$

$$\Rightarrow \boxed{A=A(T_2)}$$

$$② \quad D_0^2u_1+\omega_0^2u_1=-2D_0D_1u_0-\alpha u_0^2=-\alpha\left(A^2e^{2i\omega_0T_0}+2A\bar{A}\right)+CC$$

Particular solution of ② $\Rightarrow u_1=c_1e^{2i\omega_0T_0}+c_2A\bar{A}+CC$

$$D_0u_1=2i\omega_0c_1e^{2i\omega_0T_0} \longrightarrow D_0^2u_1=-4\omega_0^2c_1e^{2i\omega_0T_0}$$

$$\Rightarrow -4\omega_0^2c_1e^{2i\omega_0T_0}+\omega_0^2c_1e^{2i\omega_0T_0}+c_2\omega_0^2A\bar{A}=-\alpha\left(A^2e^{2i\omega_0T_0}+2A\bar{A}\right)$$

$$-4\omega_0^2 c_1 e^{2i\omega_0 T_0} + \omega_0^2 c_1 e^{2i\omega_0 T_0} + c_2 \omega_0^2 A\bar{A} = -\alpha\left(A^2 e^{2\,2i\omega_0 T_0} + 2A\bar{A}\right)$$

$$\Rightarrow -3\omega_0^2 c_1 e^{2i\omega_0 T_0} + c_2 \omega_0^2 A\bar{A} = -\alpha A^2 e^{2i\omega_0 T_0} - 2\alpha A\bar{A}$$

$$\Rightarrow -3\omega_0^2 c_1 = -\alpha A^2 \Rightarrow c_1 = \frac{\alpha A^2}{3\omega_0^2}$$

$$\Rightarrow c_2 \omega_0^2 = -2\alpha \Rightarrow c_2 = \frac{-2\alpha}{\omega_0^2}$$

$$\Rightarrow \boxed{c_1 = \frac{\alpha A^2}{3\omega_0^2} e^{2i\omega_0 T_0} - \frac{2\alpha}{\omega_0^2} A\bar{A} + CC}$$

(3) $D_0^2 u_2 + \omega_0^2 u_2 = -2D_0 D_1 u_1 - D_1^2 u_0 - 2D_0 D_2 u_0 - 2\alpha u_0 u_1$

$$u_1 = \frac{\alpha A^2}{3\omega_0^2} e^{2i\omega_0 T_0} - \frac{2\alpha}{\omega_0^2} A\bar{A} + CC \quad , \quad A = A(T_2)$$

$$u_0(T_0, T_1, T_2) = A(T_1, T_2) e^{i\omega_0 T_0} + \bar{A} e^{-i\omega_0 T_0} = u(T_0, T_2)$$

$$-2D_0 D_1 u_1 = 0 \quad ? \quad u_1 = u_1(T_0, T_2)$$

$$-D_1^2 u_0 = 0$$

$$-2D_0 D_2 u_0 = -2D_2 D_0 u_0 = -2D_2 \left(i\omega_0 A(T_2) e^{i\omega_0 T_0} + cc \right)$$

$$= -2i\omega_0 D_2 A(T_2) e^{i\omega_0 T_0} + cc$$

$$-2\lambda u_0 u_1 = -2\alpha\left(\frac{\alpha \bar{A} A^2}{3\omega_0^2} - \frac{2\alpha \bar{A} A^2}{\omega_0^2}\right) e^{i\omega_0 T_0} + NST$$

$$\begin{cases} u_1 = \dfrac{\alpha A^2}{3\omega_0^2} e^{2i\omega_0 T_0} - \dfrac{2\alpha}{\omega_0^2} A\bar{A} + CC \\ u_0(T_0, T_1, T_2) = A e^{i\omega_0 T_0} + \bar{A} e^{-i\omega_0 T_0} \end{cases}$$

$e^{-2i\omega_0 T_0}$

$$-2D_0D_2u_0 - 2\alpha u_0u_1 = 0$$

$$\frac{1}{3} - \frac{6}{3} = -\frac{5}{3}\frac{\alpha\bar{A}A^2}{\omega_0^2}$$

③ $\Rightarrow$

$$\left[-2i\omega_0 D_2A(T_2) - 2\alpha\left(\frac{\alpha\bar{A}A^2}{3\omega_0^2} - \frac{2\alpha\bar{A}A^2}{\omega_0^2}\right)\right]e^{i\omega_0T_0}$$

Eliminating S.T. $\Rightarrow$

$$\begin{cases} A(T_2) = \frac{1}{2}a(T_2)e^{i\beta(T_2)} \\ \bar{A}(T_2) = \frac{1}{2}a(T_2)e^{-i\beta(T_2)} \end{cases}$$

بدیهی است $\Rightarrow$

$$\begin{cases} a' = 0 \longrightarrow a = a_0 \\ a\beta' = -\frac{5\alpha^2}{12\omega_0^3}a^3 \xrightarrow{a_0 \neq 0} \beta' = -\frac{5\alpha^2}{12\omega_0^3}a_0^2 \end{cases}$$

$$\Rightarrow \begin{cases} a = a_0 \\ \beta = -\dfrac{5\alpha^2 a_0^2}{12\omega_0^3} T_2 + \beta_0 \end{cases} \Rightarrow A = \frac{1}{2} a e^{i\beta}$$

$$\frac{1}{2} a_0 e^{i\beta} + \frac{1}{2} a_0 e^{-i\beta} = a_0 \cos(\beta)$$

$$A = \frac{1}{2} a_0 \exp\left(-\frac{5\alpha^2 a_0^2 i}{12\omega_0^3} T_2 + i\beta_0 \right) \qquad \varepsilon^2 t$$

$$u_0 = A e^{i\omega_0 T_0} + \bar{A} e^{i\omega_0 T_0} = \frac{1}{2} a_0 \exp\left(-\frac{5\alpha^2 a_0^2 i}{12\omega_0^3} T_2 + i\beta_0 + i\omega_0 T_0 \right) + cc$$

$$\Rightarrow u_0 = a_0 \cos\left(\omega_0 t - \frac{5\alpha^2 a_0^2 \varepsilon^2 t}{12\omega_0^3} + \beta_0 \right)$$

$$u_1 = \frac{\alpha A^2}{3\omega_0^2} e^{2i\omega_0 T_0} - \frac{2\alpha}{\omega_0^2} A\bar{A} + CC$$

$$A = \frac{1}{2} a_0 e^{i\beta} \qquad \bar{A} = \frac{1}{2} a_0 e^{-i\beta}$$

$$A = \frac{1}{2} a_0 \exp\left(-\frac{5\alpha^2 a_0^2 i T_2}{12\omega_0^3} + i\beta_0 \right)$$

$$u_1 = \frac{\alpha a_0^2}{3\times 4\omega_0^2} \exp\left(2i\omega_0 T_0 - \frac{5\alpha^2 a_0^2 i T_2}{6\omega_0^3} + 2i\beta_0 \right) - \frac{2\alpha}{\omega_0^2}\left(\frac{1}{4} a_0^2\right) + CC$$

$$\Rightarrow u_1 = \frac{\alpha a_0^2}{6\omega_0^2} \cos\left(2\omega_0 t - \frac{5\alpha^2 a_0^2 \varepsilon^2 t}{6\omega_0^3} + 2\beta_0 \right) - \frac{\alpha a_0^2}{2\omega_0^2}$$

$$u = u_0 + \varepsilon u_1$$

$$u_0 = a_0 \cos\left(\omega_0 t - \frac{5\alpha^2 a_0^2 \varepsilon^2 t}{12\omega_0^3} + \beta_0\right)$$

$$u_1 = \frac{\alpha a_0^2}{6\omega_0^2} \cos\left(2\omega_0 t - \frac{5\alpha^2 a_0^2 \varepsilon^2 t}{6\omega_0^3} + 2\beta_0\right) - \frac{\alpha a_0^2}{2\omega_0^2}$$

با تعریف ω به صورت زیر

$$, \omega = \omega_0 \left[1 - \frac{5\alpha^2 a_0^2 \varepsilon^2}{12\omega_0^4}\right]$$

$$u = a_0 \cos(\omega t + \beta_0) + \frac{\varepsilon \alpha a_0^2}{2\omega_0^2}\left[-1 + \frac{1}{3}\cos(2\omega t + 2\beta_0)\right]$$

با تقریب ω به صورت زیر

$$\omega = \omega_0 \left[1 - \frac{5\alpha^2 a_0^2 \varepsilon^2}{12\omega_0^4} \right]$$

$$u = a_0 \cos(\omega t + \beta_0) + \frac{\varepsilon \alpha a_0^2}{2\omega_0^2} \left[-1 + \frac{1}{3}\cos(2\omega t + 2\beta_0) \right]$$

drift zero frequency

- قرینس نوسانات یک ناشی از دامنه (a_0) نوسانات است.
- مستقل از علامت α، قرینس همواره کاهش می‌یابد. (softening)
- در جواب u یک عدم تقارن $\left(\frac{\varepsilon \alpha a_0^2}{2\omega_0^2}\right)$ هم این‌گونه نوسان به آن drift می‌گوییم.