



دانشگاه سمنان

ارتعاشات غیر خطی

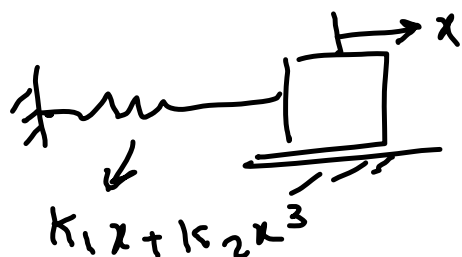
Perturbation Theory Quadratic Nonlinearity

دکتر امین نیکوبین

دانشگاه سمنان، دانشکده مهندسی مکانیک

anikoobin@semnan.ac.ir

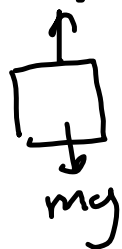
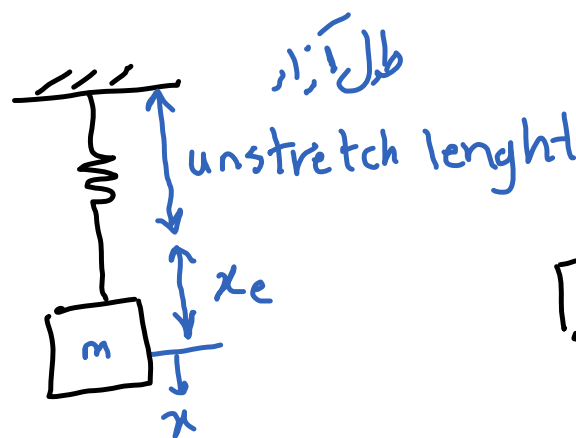
EX.



معادله

$$\ddot{x} + x + \alpha x^3 = 0$$

ارتعاشات افقی سیستم جرم و فنر



$$m\ddot{x} + k_1 x + k_2 x^3 = mg$$

معادله بی بعدی
معمول

$$\ddot{x} + x + \alpha x^3 = \beta \quad (1)$$

نقطه تعادل: $\ddot{x} = 0 \Rightarrow x_e + \alpha x_e^3 = \beta \quad (2)$

نکته: فصل تعاد، تعادل را در نظر بگیرید

$$x(t) = x_e + u(t) \rightarrow \ddot{x} = \ddot{u}$$

$$\textcircled{1} \Rightarrow \ddot{u} + x_e + u + \alpha(x_e + u)^3 = \beta$$

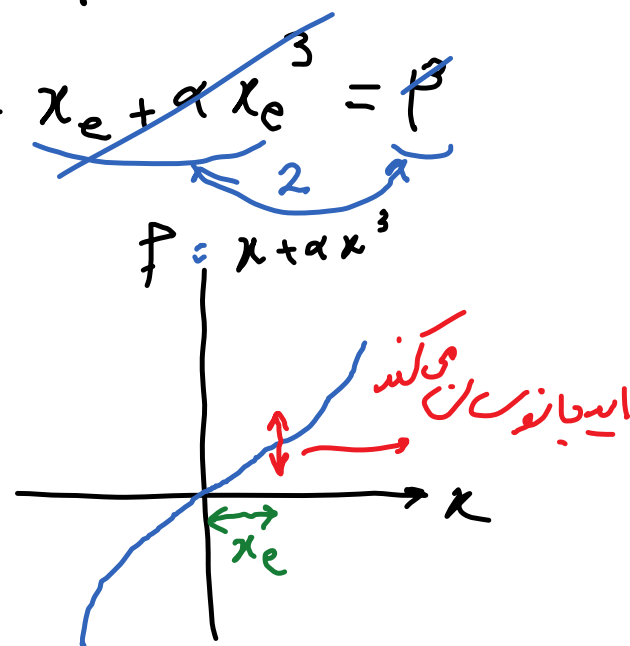
$$\Rightarrow \ddot{u} + (1 + 3\alpha x_e^2)u + 3\alpha x_e u^2 + \alpha u^3 + \cancel{x_e + \alpha x_e^3} = \beta$$

$$\Rightarrow \ddot{u} + \alpha_1 u + \alpha_2 u^2 + \alpha_3 u^3 = 0$$

$$\text{Scaling } u = \varepsilon v \rightarrow \ddot{u} = \varepsilon \ddot{v}$$

$$\Rightarrow \varepsilon \ddot{v} + \alpha_1 \varepsilon v + \alpha_2 \varepsilon^2 v^2 + \alpha_3 \varepsilon^3 v^3 = 0$$

$$\Rightarrow \ddot{v} + \alpha_1 v + \alpha_2 \varepsilon v^2 + \alpha_3 \varepsilon^2 v^3 = 0$$





$$\ddot{v} + \alpha_1 v + \alpha_2 \varepsilon v^2 + \alpha_3 \varepsilon^2 v^3 = 0$$

به خاطر وجود ترم $\alpha_2 \varepsilon v^2$ ؛ second order ex.

$$v = v_0 + \varepsilon v_1 + \varepsilon^2 v_2 + \dots$$

$$u = a_0 \cos(\omega t + \beta_0) - \frac{\varepsilon a_0^2 \alpha_2}{2 \omega_0^2} \left[1 - \frac{1}{3} \cos(2\omega t + 2\beta_0) \right]$$

$$\omega = \omega_0 \left[1 + \frac{4 \alpha_3 \omega_0^2 - 10 \alpha_2^2}{24 \omega_0^4} (\varepsilon a_0)^2 \right]$$

$\alpha_2 \varepsilon v^2$ (yellow circle)
 $\alpha_3 \varepsilon^2 v^3$ (blue circle)

Harmonic balance method

$$\ddot{u} + \omega_0^2 u + \alpha_2 u^2 = 0$$

$$u = A_1 \cos \varphi, \quad \varphi = \omega t + \beta$$

$$\Rightarrow -(\omega^2 - \omega_0^2) A_1 \cos \varphi + \alpha_2 A_1^2 \cos^2 \varphi = 0$$

$$\Rightarrow (\omega_0^2 - \omega^2) A_1 \cos \varphi + \frac{\alpha_2 A_1^2}{2} + \frac{\alpha_2 A_1^2}{2} \cos 2\varphi = 0$$

$$A_1 \approx O(\epsilon) \rightarrow A_1^2 \approx 0$$

$$(\omega_0^2 - \omega^2) A_1 \cos \varphi = 0 \Rightarrow \omega = \omega_0 \rightarrow u = A_1 \cos(\omega_0 t + \beta)$$

که همان جواب خطی را می‌دهد، عملاً غایب می‌باشد.

نیم‌های فرکانس بالا مدبر می‌شود

$$\frac{1}{2} + \frac{1}{2} \cos 2\varphi$$

پس به تدریج مرتبه بالاتر را هم در نظر بگیریم.

$$u = A_0 + A_1 \cos \varphi + A_2 \cos 2\varphi, \quad \varphi = \omega t + \beta_0$$

$$\hookrightarrow \dot{u} \rightarrow \ddot{u} \rightarrow$$

$$\textcircled{1} \text{ Constant: } \cancel{\alpha_2 A_0^2} + \omega_0^2 A_0 + \frac{1}{2} \alpha_2 (\cancel{A_1^2} + \cancel{A_2^2}) = 0$$

$$\textcircled{2} \cos \varphi: A_1 (\omega_0^2 - \omega^2) + 2\alpha_2 A_0 A_1 + \alpha_2 A_1 A_2 = 0$$

$$\textcircled{3} \cos 2\varphi: A_2 (-4\omega^2 + \omega_0^2) + \frac{\alpha_2 A_1^2}{2} + 2\alpha_2 A_0 A_2 = 0$$

$$\textcircled{1} \rightarrow \omega_0^2 A_0 + \frac{\alpha_2 A_1^2}{2} = 0 \rightarrow$$

ملته: $A_2 \ll A_1$ و $A_0 \approx 0(\epsilon)$

$$A_0 = -\frac{\alpha_2}{2\omega_0^2} A_1^2$$



$$\textcircled{2} A_1(\omega_0^2 - \omega^2) + 2\alpha_2 A_0 A_1 + \alpha_2 A_1 A_2 = 0$$

$$\Rightarrow (\omega_0^2 - \omega^2) + 2\alpha_2 A_0 + \alpha_2 A_2 = 0$$

$$A_0 = -\frac{\alpha_2}{2\omega_0^2} A_1^2$$

$$\Rightarrow (\omega_0^2 - \omega^2) - \frac{\alpha_2^2 A_1^2}{\omega_0^2} + \alpha_2 A_2 = 0$$

$$\textcircled{3} \rightarrow A_2(-4\omega^2 + \omega_0^2) + \frac{\alpha_2 A_1^2}{2} + 2\alpha_2 A_0 A_2 = 0$$

$$\rightarrow A_2(-4\omega^2 + \omega_0^2) + \frac{\alpha_2 A_1^2}{2} - \frac{\alpha_2^2}{\omega_0^2} A_2 A_1^2 = 0$$

$$(A_2 \ll A_1)$$

$$\begin{cases} A_2 = \frac{\alpha_2}{6\omega_0^2} A_1^2 \\ \omega^2 = \omega_0^2 - \frac{5\alpha_2^2}{6\omega_0^2} A_1^2 \end{cases}$$



$$u = \frac{\alpha_2 A_1^2}{2\omega_0^2} + A_1 \cos(\omega t + \beta_0) + \frac{1}{6} \frac{\alpha_2}{\omega_0^2} A_1^2 \cos(2\omega t + 2\beta_0)$$

$$\Rightarrow u = A_1 \cos(\omega t + \beta_0) - \frac{A_1^2 \alpha_2}{2\omega_0^2} \left[1 - \frac{1}{3} \cos(2\omega t + 2\beta_0) \right]$$

$$\omega^2 = \omega_0^2 - \frac{5\alpha_2^2}{6\omega_0^2} A_1^2 \Rightarrow \omega = \omega_0 \left[1 - \frac{5\alpha_2^2}{6\omega_0^4} A_1^2 \right]^{\frac{1}{2}}$$

$$\Rightarrow \omega = \omega_0 \left[1 - \frac{5\alpha_2^2 A_1^2}{12\omega_0^4} \right]$$

EX. $\ddot{u} - u + u^4 = 0$

صرفاً ہم نوں سے این کہیم اے عمل $u=1$ مور مطالعہ وادہیم

$$\ddot{u} = 0 \rightarrow u(u^3 - 1) = 0 = u(u-1)(u^2+u+1) \rightarrow \begin{matrix} u=0 \\ u=1 \end{matrix}$$

باضعہ مقدر $u = 1+v \Rightarrow \ddot{v} - 1 - v + (1+v)^4 = 0$

$$\Rightarrow \ddot{v} + 3v + 6v^2 + 4v^3 + v^4 = 0$$

از انبارہ v کو صیقلے v^4 صرف نظر می کشیم

$$\ddot{v} + 3v + 6v^2 + 4v^3 = 0$$



$$\ddot{v} + 3v + 6v^2 + 4v^3 = 0$$

$$, v' = \frac{dv}{d\tau}$$

$$\text{Transformation} \rightarrow \tau = \omega_0 t = \sqrt{3} t \rightarrow d\tau = \sqrt{3} dt$$

$$\dot{v} = \frac{dv}{dt} = \frac{dv}{d\tau} \frac{d\tau}{dt} = \sqrt{3} v', \quad \ddot{v} = \sqrt{3} \sqrt{3} v'' = 3v''$$

$$\Rightarrow 3v'' + 3v + 6v^2 + 4v^3 = 0 \Rightarrow \boxed{v'' + v + 2v^2 + \frac{4}{3}v^3 = 0}$$

با این تغییر متغیر و ناسنس $\omega_0 = 1$ شد.

$$\Rightarrow \alpha_2 = 2, \quad \alpha_3 = \frac{4}{3}$$

$$\Rightarrow v = a_0 \cos(\omega\tau + \beta_0) + \frac{1}{3}a_0^2 [\cos(2\omega\tau + 2\beta_0) - 3]$$

$$\omega = 1 - \frac{7}{6}a_0^2$$



ارتعاشات غیر خطی، Quadratic Nonlinearity

دکتر امین نیکوبین