کنترل اتوماتیک

مروری بر تبدیل لاپلاس

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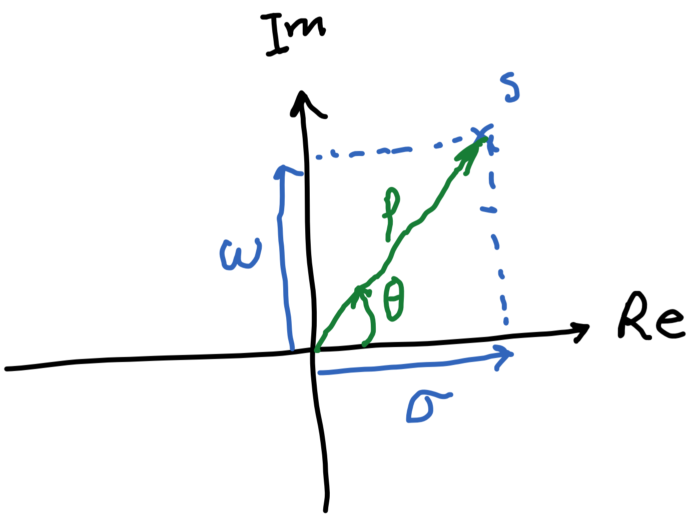
$$s = \sigma + j\omega = \rho e^{j\theta}$$

عدد موهومی



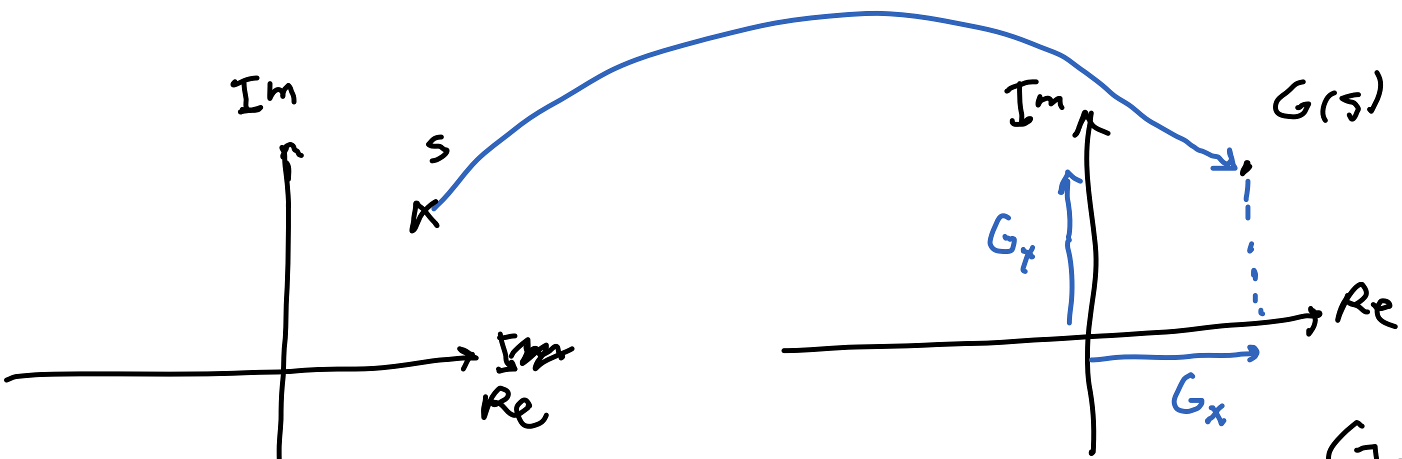
$$e^{j\theta} = \cos\theta + j\sin\theta$$

$$\rho = \sqrt{\sigma^2 + \omega^2}$$

$$\theta = tg^{-1}\frac{\omega}{\sigma}$$

تابع موهومی $G(s)$

مثال: $G(s) = \dfrac{1}{s+1}$, $\dfrac{s+1}{s^2+s+2}$,



$G(s) = G_x + j\,G_y$,

$G_x = Re\,[G(s)]$

$G_y = Im\,[G(s)]$

بخش حقیقی و موهومی تابع $G(s)$ زیر را بدست آورید

$$G(s) = \frac{s+1}{s+2} = \frac{\sigma + j\omega + 1}{\sigma + j\omega + 2} =$$

$$G(s) = \frac{(\sigma+1)+j\omega}{\sigma+2+j\omega} \times \frac{(\sigma+2)-j\omega}{(\sigma+2)-j\omega} = \frac{(\sigma+1)(\sigma+2) - j\omega(\sigma+1) + j\omega(\sigma+2) + \omega^2}{(\sigma+2)^2 + \omega^2}$$

$$j^2 = -1 \,, \quad j^3 = -j \,, \quad j^4 = 1$$

$$\Rightarrow \frac{(\sigma+1)(\sigma+2) + \omega^2 + j\left[-\omega(\sigma+1) + \omega(\sigma+2)\right]}{(\sigma+2)^2 + \omega^2}$$

$$G(s) = \frac{\left[(\sigma+1)(\sigma+2)+\omega^2\right] + j\left[\overbrace{-\omega(\sigma+1)+\omega(\sigma+2)}^{\omega}\right]}{(\sigma+2)^2+\omega^2}$$

$$\Rightarrow \begin{cases} Re\left[G(s)\right] = \dfrac{(\sigma+1)(\sigma+2)+\omega^2}{(\sigma+2)^2+\omega^2} \\ \\ Im\left[\ \text{"}\ \right] = \dfrac{\omega}{(\sigma+2)^2+\omega^2} \end{cases}$$

صفر و قطب یک تابع موهومی

$$G(s) = \frac{s+2}{s+1}$$

صفر تابع $G(s)$ zero : $G(s^*) = 0 \Rightarrow s^* = -2$ صفر تابع $z_1 = -1$

قطب تابع Pole : $G(s^*) = \infty \Rightarrow s^* = -1$ قطب تابع $p_1 = -1$

$$G(s) = \frac{s+1}{s(s+2)^3} = \frac{s+1}{s(s+2)(s+2)(s+2)}$$

یک صفر دارد و چهار قطب

$$z_1 = -1$$

$$p_1 = 0$$

$$p_2 = p_3 = p_4 = -2$$

$$G(s) = \frac{(s+1)(2s+1)}{s^2(s^2+s+2)}$$

$$as^2+bs+c=0$$

$$s_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$z_1 = -1, \quad z_2 = -\frac{1}{2}$$

$$p_1 = p_2 = 0$$

$$s^2+s+2=0 \longrightarrow p_{3,4} = \frac{-1 \pm \sqrt{1-8}}{2} = \frac{-1}{2} \pm \frac{\sqrt{7}j}{2}$$

قضیه اویلر

$$\begin{cases} e^{j\theta} = \cos\theta + j\sin\theta & (1) \\ e^{-j\theta} = \cos\theta - j\sin\theta & (2) \end{cases}$$

بجای Sin و Cos بصورت
سری تیلور

$$(1)+(2) \rightarrow \cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

$$(1)-(2) \rightarrow \sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

تبدیل لاپلاس

$f(t)$ برای $t<0$ ، $f(t)=0$

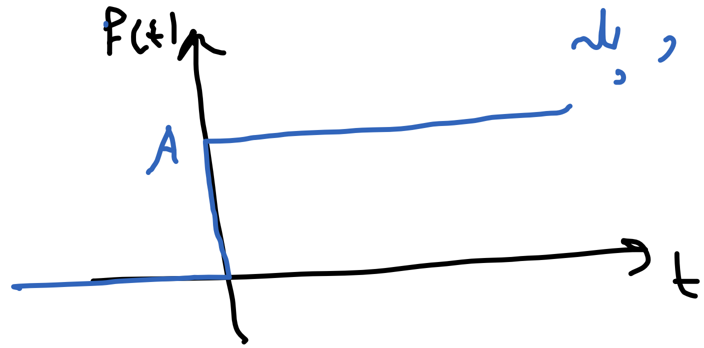
s عدد مختلط باشد

$$f(t) \xrightarrow{\mathcal{L}} F(s)$$

$$\mathcal{L}\left[f(t)\right] = F(s) = \int_0^{\infty} e^{-st} f(t)\,dt$$

مثال: step و پله

$$f(t) = \begin{cases} 0 & t < 0 \\ A & t \geq 0 \end{cases}$$



$$F(s) = \int_0^\infty A e^{-st}\, dt = A\left[\frac{e^{-st}}{-s}\right]_0^\infty = A\left[0 - \left(\frac{-1}{s}\right)\right] = \frac{A}{s}$$

$$f(t) = Ae^{-at} \longrightarrow F(s) = \frac{A}{s+a}$$

$$f(t) = At \longrightarrow F(s) = \frac{A}{s^2}$$

Ramp شیب

$$\mathcal{L}\left[A\sin\omega t\right] = \frac{A\omega}{s^2+\omega^2}$$

$$\mathcal{L}\left[A\cos\omega t\right] = \frac{As}{s^2+\omega^2}$$

$$\mathcal{L}\left[\frac{t^2}{2}\right] = \frac{1}{s^3} \quad \text{و} \quad \mathcal{L}\left[\frac{t^3}{6}\right] = \frac{1}{s^4}$$

سی

$$\mathcal{L}(\dot{x}(t)) = sX(s) - x(0)$$

* مهم

$$\mathcal{L}[\ddot{x}(t)] = s^2 X(s) - s\,x(0) - \dot{x}(0)$$

$$\mathcal{L}\left[\int f(t)\,dt\right] = \frac{F(s)}{s}$$

S عامل مشتق گیری

$\frac{1}{S}$ عامل انتگرال گیری.

$F(s)$ را می‌خواهیم در حوزه زمان از آن مشتق بگیریم

$$F(s) \xrightarrow{\mathcal{L}^{-1}} f(t) \longrightarrow \dot{f}(t)$$

$$F(s) \longrightarrow s\,F(s)$$

$$\frac{1}{s} F(s) \equiv \int f(t)\,dt$$

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

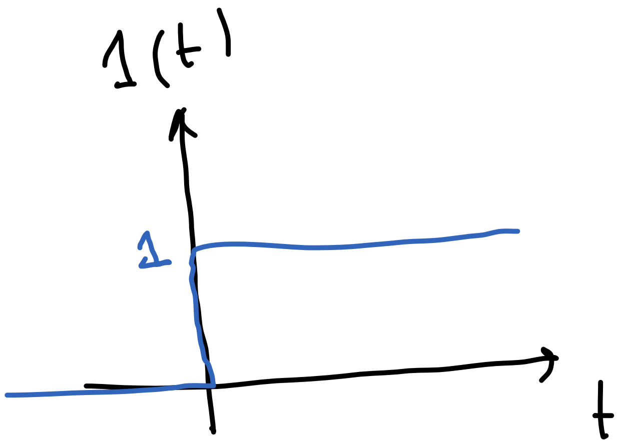
بدست آوردن مقدار نهایی یک تابع ، خطای حالت ماندگار

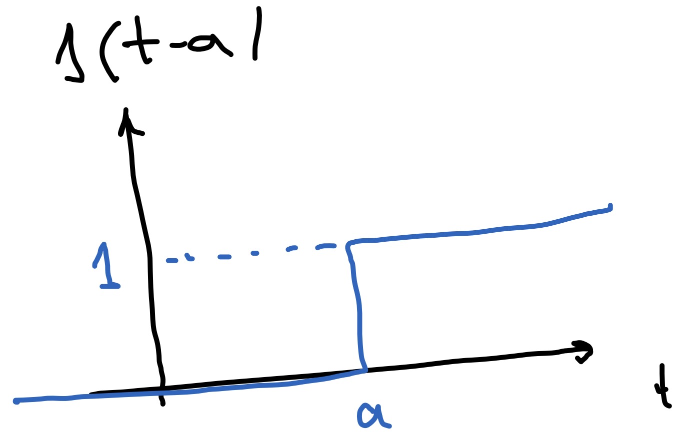
$$\mathcal{L}\left[e^{-at} f(t)\right] = F(s+a)$$

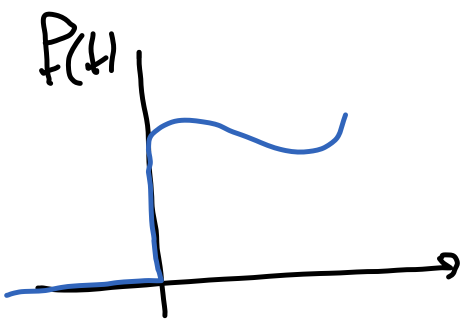
$$e^{-at}\sin\omega t = \frac{A\omega}{(s+a)^2+\omega^2}$$

$$e^{-at}\cos\omega t = \frac{A(s+a)}{(s+a)^2+\omega^2}$$

$$\mathcal{L}\left[f(t-a)\, 1(t-a) \right]$$



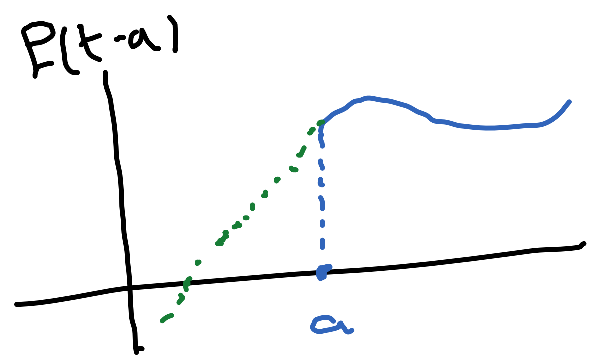


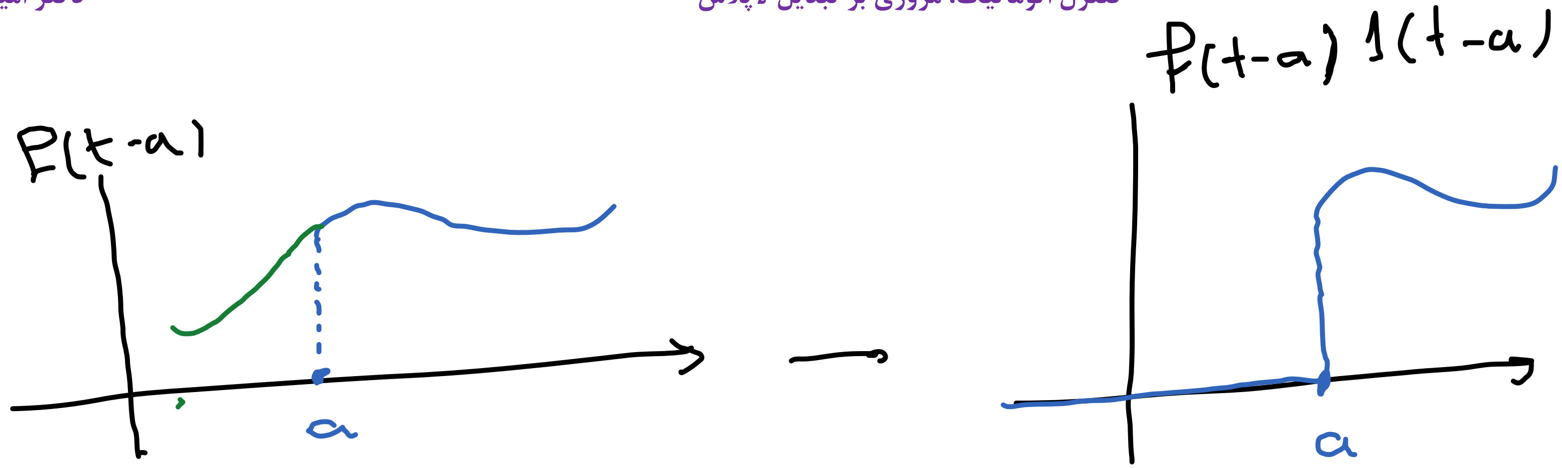


تأخیر به اندازه a

$\xrightarrow{\hspace{2cm}}$

$f(t-a)$



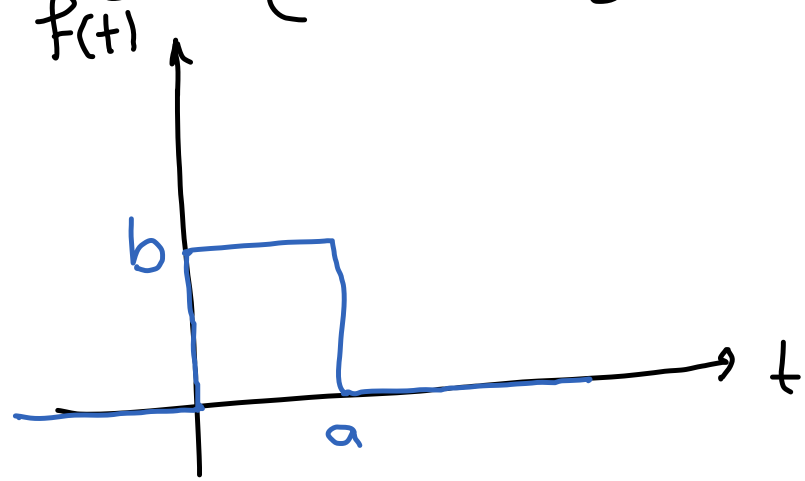


$$\mathcal{L}\left[f(t-a)\,1(t-a) \right] = e^{-as} F(s)$$

مثال: تبدیل لاپلاس تابع پالس زیر را بدست آورید

$$f = f_1 + f_2$$

$$f(t) = \begin{cases} 0 & t < 0 \\ b & 0 \leq t < a \\ 0 & t > a \end{cases}$$



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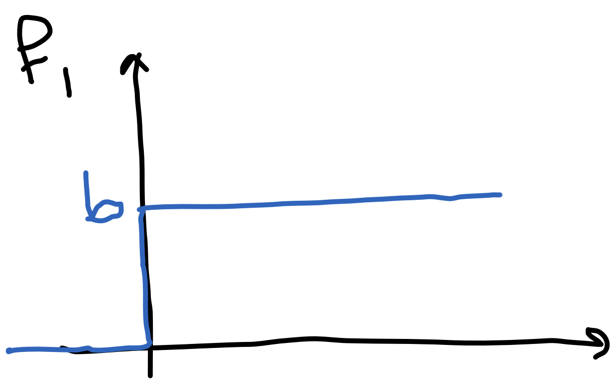

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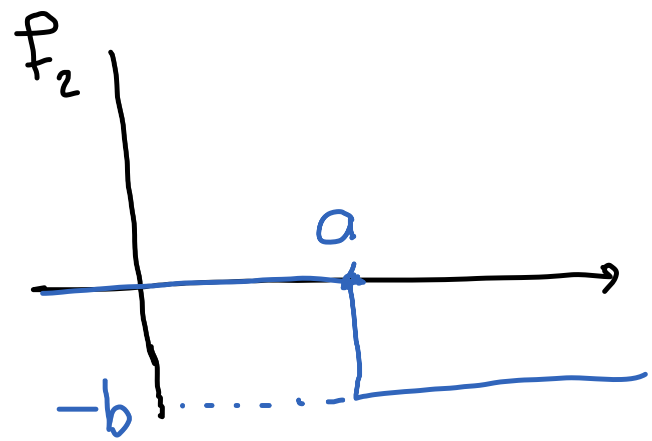

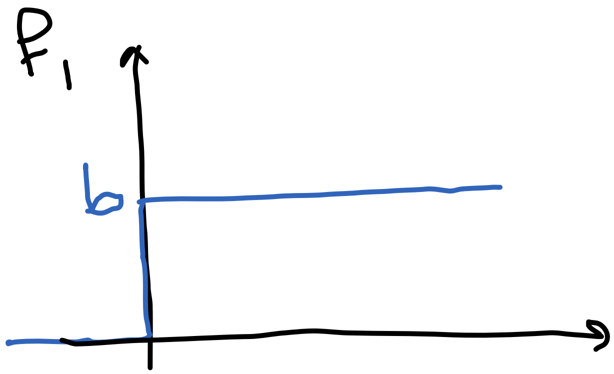

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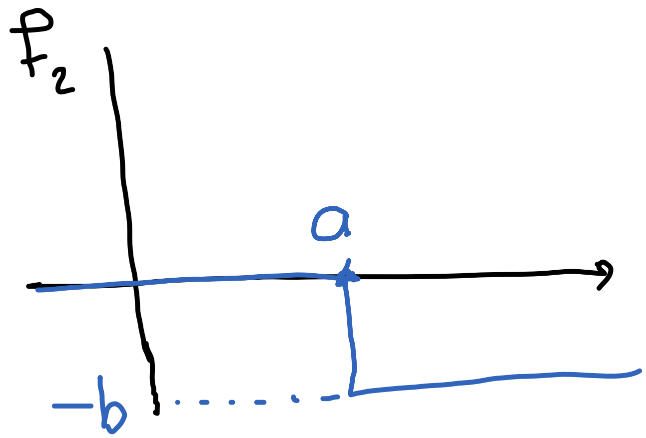


$f_1(t) = b\,1(t)$

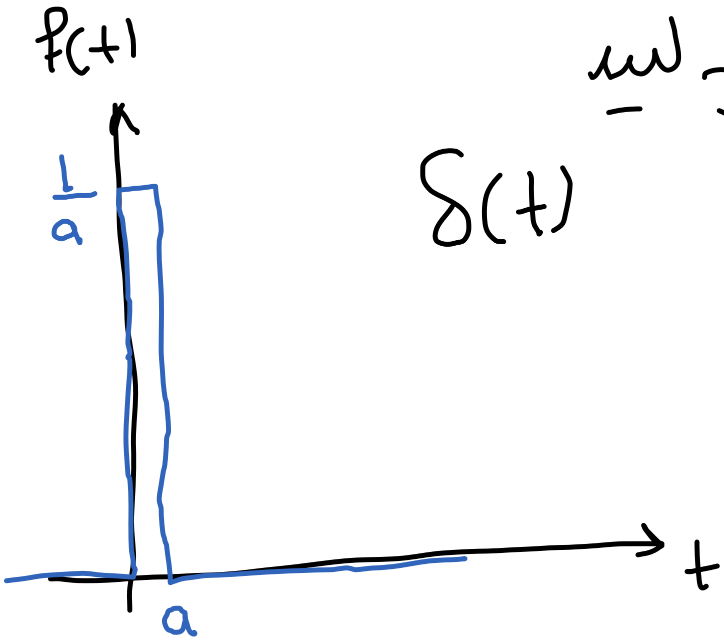
$f_2 = -b\,1(t-a)$

$F_1(s) = \frac{b}{s}$

$F_2(s) = \frac{-b}{s} e^{-as}$

$F(s) = \frac{b}{s} - \frac{b}{s} e^{-as}$

مثال: تبدیل لاپلاس تابع ضربه واحد را حساب کنید



$\delta(t)$

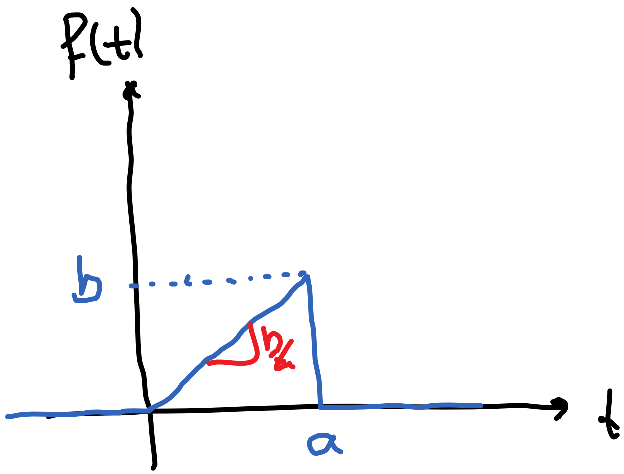
در مثال قبلی $b \to \frac{1}{a}$ و $a \to 0$

$$F(s) = \frac{b}{s} - \frac{b}{s} e^{-as}$$

$$\Rightarrow F(s) = \underset{a \to 0}{\frac{1}{as} - \frac{e^{-as}}{as}} = \underset{a \to 0}{\frac{1 - e^{-as}}{as}} \xRightarrow{\text{هوپیتال}} \underset{a \to 0}{\frac{\cancel{s} e^{-as}}{\cancel{s}}} = e^{-as} = 1$$

$$\mathcal{L}[\delta(t)] = 1$$

مثال: تبدیل لاپلاس تابع زیر را محاسبه کنید



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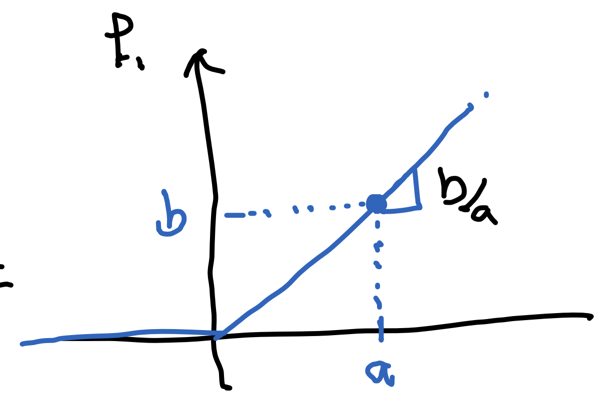

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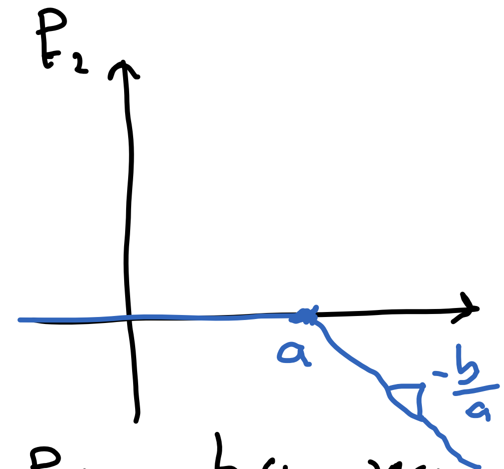
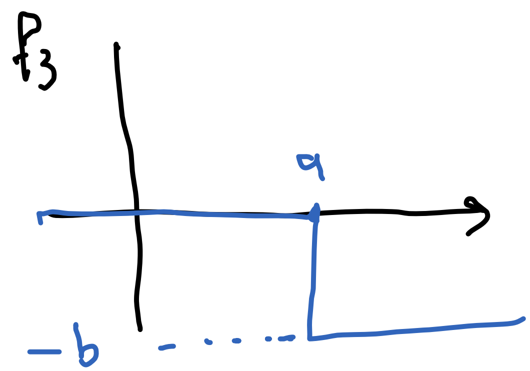


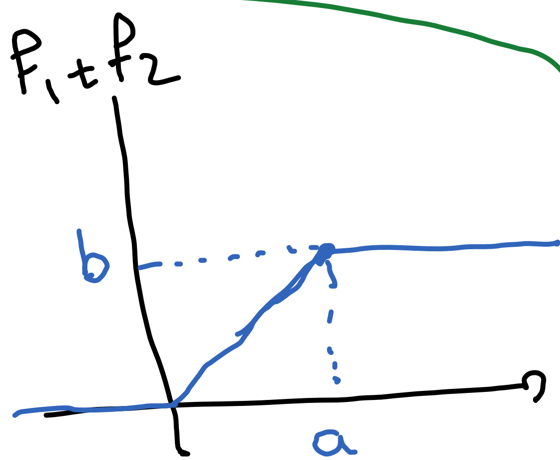
$$f_1(t) = \frac{b}{a}t$$

$$f_2(t) = -\frac{b}{a}(t-a)1(t-a)$$

+



$$f_3 = -b\,1(t-a)$$



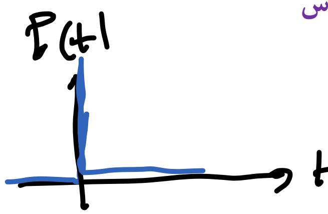
$$f_1(t) = \frac{b}{a}t \longrightarrow F_1(s) = \frac{b}{a}\frac{1}{s^2}$$

$$f_2(t) = -\frac{b}{a}(t-a)\,1(t-a) \longrightarrow F_2(s) = -\frac{b}{a}\frac{1}{s^2}e^{-as}$$

$$f_3 = -b\,1(t-a) \longrightarrow F_3(s) = -b\frac{1}{s}e^{-as}$$

$$F(s) = F_1 + F_2 + F_3$$

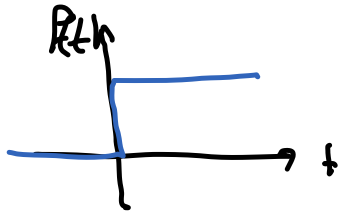
$f(t)=\delta(t)$, impulse ضربه



$F(s)=1$

$\frac{d}{dt}$ $\int$ $\frac{1}{s}$ $\times s$

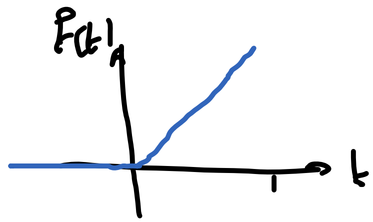
$f(t)=1$, step پله



$F(s)=\frac{1}{s}$

$\frac{d}{dt}$ $\int$ $\frac{1}{s}$ $\times s$

$f(t)=t$, ramp شیب



$F(s)=\frac{1}{s^2}$

$\frac{d}{dt}$ $\int$ $\frac{1}{s}$ $\times s$

$f(t)=\frac{t^2}{2}$, سهمی

$F(s)=\frac{1}{s^3}$

عکس تبدیل لاپلاس:

بسط: کسرهای جزئی

با روش قطعه‌های مجزا

$$F(s)=\frac{A(s)}{B(s)}=\frac{k\left(s^{m}+b_{m-1}s^{m-1}+\cdots+b_{0}\right)}{s^{n}+a_{n-1}s^{n-1}+\cdots+a_{0}} \quad , \quad n \geq m$$

$$F(s)=\frac{A(s)}{B(s)}=\frac{k\left(s^{m}+b_{m-1}s^{m-1}+\cdots+b_{0}\right)}{(s+p_{1})(s+p_{2})\cdots(s+p_{n})}=\frac{a_1}{s+p_1}+\frac{a_2}{s+p_2}+\cdots+\frac{a_n}{s+p_n}$$

$$a_i=\lim_{s\to -p_i}\left[(s+p_i)\,F(s)\right]$$

$$G(s) = \frac{s+3}{s(s^2+3s+2)} = \frac{s+3}{s(s+1)(s+2)} = \frac{a_1}{s} + \frac{a_2}{s+1} + \frac{a_3}{s+2}$$

$$a_1 = \lim_{s \to 0} sG(s) = \lim_{s \to 0} \cancel{s}\frac{s+3}{\cancel{s}(s^2+3s+2)} = \frac{3}{2}$$

$$a_2 = \lim (s+1)G(s) = \lim \cancel{(s+1)}\frac{s+3}{s\cancel{(s+1)}(s+2)} = \frac{-1+3}{(-1)(-1+2)} = \frac{2}{-1}$$

$$a_3 = \lim (s+2)G(s) = \xrightarrow{s \to -1} \frac{-2+3}{(-2)(-2+1)} = \frac{1}{2}$$

$$G(s) = \frac{a_1}{s} + \frac{a_2}{s+1} + \frac{a_3}{s+2} = \frac{1.5}{s} + \frac{-2}{s+1} + \frac{0.5}{s+2}$$

$$\Rightarrow g(t) = \mathcal{L}^{-1}\left(G(s)\right) = 1.5 - 2e^{-t} + 0.5e^{-2t}$$

تعریف تابع تبدیل در MATLAB

$$G(s) = \frac{s+3}{s(s^2+3s+2)}$$

$$G(s) = \frac{5(s+3)}{s^3+3s^2+2s+0} \Rightarrow G(s) = \frac{num}{den} = \frac{[1 \quad 3]}{[1 \quad 3 \quad 2 \quad 0]}$$

G = tf(num, den)

G = zpk([-3], [0 -1 -2])

مثال: کسرهای جزئی

$$G_2(s) = 2s + 1 + \frac{s+3}{s(s^2+3s+2)}$$

residu(num, den)

قطب تکراری

$$F(s) = \frac{A(s)}{(s+p_1)\cdots(s+p_k)^r\cdots(s+p_n)}$$

$$F(s) = \frac{a_1}{s+p_1} + \cdots + \frac{c_1}{s+p_k} + \frac{c_2}{(s+p_k)^2} + \cdots + \frac{c_r}{(s+p_k)^r} + \cdots \frac{a_n}{s+p_n}$$

$$c_{r-J} = \frac{1}{J!}\frac{d^J}{ds^J}\left[(s+p_k)^r F(s)\right]_{s=-p_k} \quad J = 0, 1, \cdots r-1$$

C_3 ، $j=0$ $\rightarrow$ $C_3 = \checkmark$ ،
$r=3$

$C_2 = \frac{1}{1!}\frac{d}{ds}[\quad " \quad]$
$j=1$

$$F(s) = \frac{1}{s(s+2)(s+1)^3} = \frac{a_1}{s} + \frac{a_2}{s+2} + \frac{c_3}{(s+1)^3} + \frac{c_2}{(s+1)^2} + \frac{c_1}{(s+1)}$$

$$a_1 = \lim_{s \to 0} s F(s) = \frac{1}{2} \quad , \quad a_2 = \lim_{s \to -2} (s+2) F(s) = \frac{1}{-2(-2+1)^3} = \frac{1}{2}$$

$$c_3 = \lim_{s \to -1} (s+1)^3 F(s) = \lim_{s \to -1} \left[\frac{1}{s(s+2)} \right] = \frac{1}{-1(-1+2)} = -1$$

$$c_2 = \frac{d}{ds} \left[\frac{1}{s^2 + 2s} \right]_{s \to -1} = \frac{0 - (2s+2)(1)}{(s^2+2s)^2} = 0$$

$$C_1 = \frac{1}{2!}\frac{d^2}{ds^2}\left[\frac{1}{s(s+2)}\right] = \frac{1}{2}\frac{d}{ds}\left[\frac{-(2s+2)}{(s^2+2s)^2}\right]$$

$$s \to -1$$

$$= \ldots = -1$$

$$F(s) = \frac{0.5}{s} + \frac{0.5}{s+2} + \frac{-1}{(s+1)^3} + \frac{0}{(s+1)^2} + \frac{-1}{s+1}$$

$$\Rightarrow f(t) = 0.5 + 0.5e^{-2t} - \frac{t^2 e^{-t}}{2} - e^{-t}$$

بسط به کسرهای جزئی برای حالت قطب موهومی

$$F(s) = \frac{2s+12}{s^2+2s+5}$$

$$= \frac{2(s+1)+5(2)}{(s+1)^2+2^2}$$

$$= \frac{2(s+1)}{(s+1)^2+2^2} + \frac{5(2)}{(s+1)^2+2^2}$$

$$\Rightarrow f(t) = 2e^{-t}\cos 2t + 5e^{-t}\sin(2t)$$

$$Ae^{-at}\sin\omega t = \frac{A\omega}{(s+a)^2+\omega^2}$$

$$Ae^{-at}\cos\omega t = \frac{A(s+a)}{(s+a)^2+\omega^2}$$

معادله دیفرانسیل زیر را با استفاده از روش تبدیل لاپلاس حل کنید.

$$\mathcal{L}\left(\ddot{x}+3\dot{x}+2x=u(t)\right) \quad , \quad x(0)=a,\ \dot{x}(0)=b,\ u(t)=5\delta(t)$$

$$s^2X(s)-sx(0)-\dot{x}(0)+3\left(sX(s)-x(0)\right)+2X(s)=U(s)$$

$$\Rightarrow X(s)\left[\overbrace{s^2+3s+2}^{A(s)}\right]=\overbrace{as+b+3a}^{W(s)}+U(s)$$

$$X(s)=\frac{W(s)}{A(s)}+\frac{U(s)}{A(s)} \Rightarrow x(t)=\mathcal{L}^{-1}\left[\frac{W}{A}\right]+\mathcal{L}^{-1}\left[\frac{U}{A}\right]$$

$$\rightarrow x(t)=x_n(t)+x_p(t)=x_{zi}+x_{zs}$$

$$x_h = x_{zi} = \mathcal{L}^{-1}\left[\frac{W}{A}\right] = \mathcal{L}^{-1}\left[\frac{as+b+3a}{s^2+3s+2}\right]$$

$$x_p = x_{zs} = \mathcal{L}^{-1}\left[\frac{U}{A}\right] = \mathcal{L}^{-1}\left[\frac{U(s)}{s^2+3s+2}\right]$$

$$U(s) = 5$$

$$x_{zs} = L^{-1}\frac{5}{s^2+3s+2} = L^{-1}\left[\frac{5}{s+1} - \frac{5}{s+2}\right] = 5(e^{-t} - e^{-2t})$$

$$x_{zi} = L^{-1}\frac{as+b+3a}{s^2+3s+2} = L^{-1}\left[\frac{2a+b}{s+1} - \frac{a+b}{s+2}\right] = (2a+b)e^{-t} - (a+b)e^{-2t})$$

$$x = x_{zs} + x_{zi}$$