کنترل اتوماتیک

تحلیل پاسخ گذرا و ماندگار سیستمهای خطی

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سیستم‌های مرتبه اول

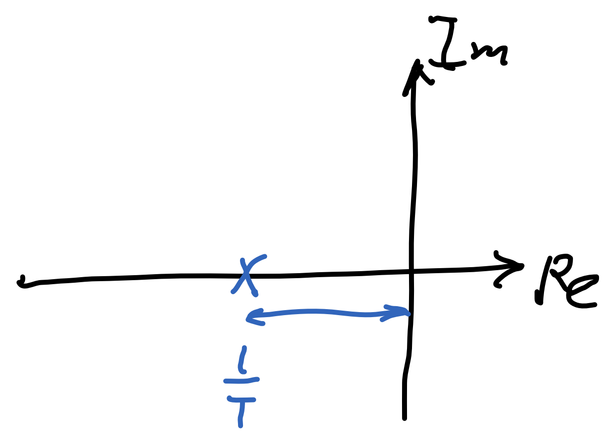
$$R(s) \longrightarrow \boxed{\frac{1}{Ts+1}} \longrightarrow C(s)$$

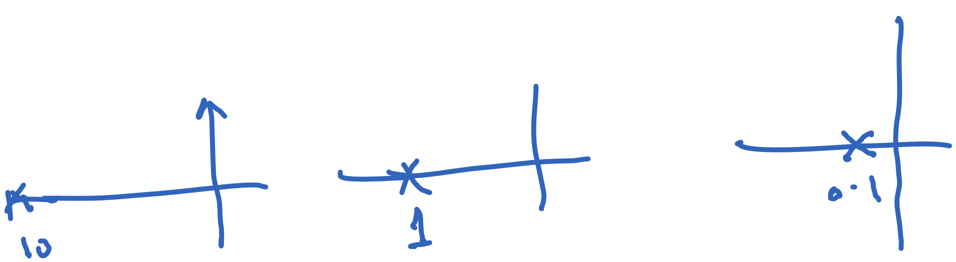
time constant ثابت زمانی T

پاسخ به ورودی پله: $R(s) = \frac{1}{s}$

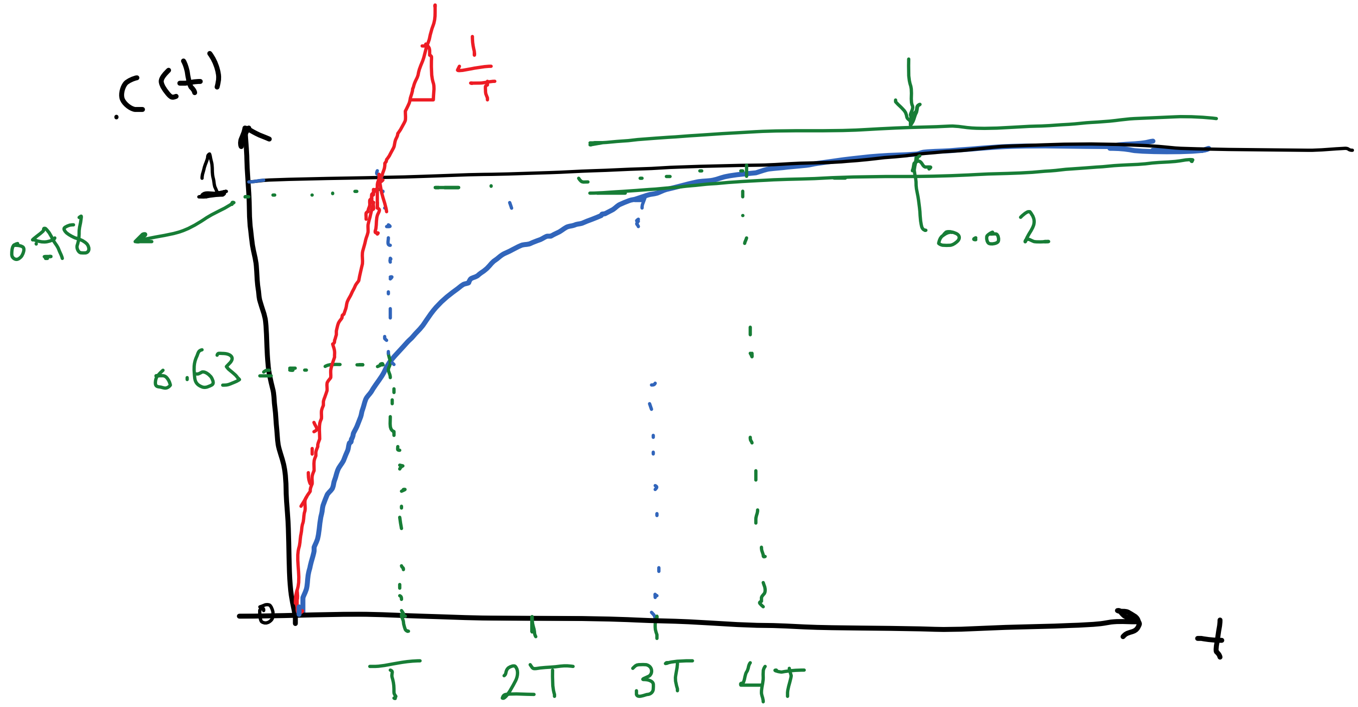
$$C(s) = \frac{1}{Ts+1} \times \frac{1}{s} = \frac{a_1}{Ts+1} + \frac{a_2}{s} = \frac{-T}{Ts+1} + \frac{1}{s} = \frac{-1}{s+\frac{1}{T}} + \frac{1}{s}$$

$$\Rightarrow c(t) = \mathcal{L}^{-1}[C(s)] = 1 - e^{-\frac{t}{T}}$$





$T = 0.1$ $T = 1$ $T = 10$



$$\frac{dc(t)}{dt} = \frac{1}{T}e^{-\frac{t}{T}}$$

$$\left.\frac{dc(t)}{dt}\right|_{t=0} = \frac{1}{T}$$

$$c = 1 - e^{-\frac{t}{T}} \Rightarrow \text{at } t = T \Rightarrow c(t) = 1 - e^{-1} = 0.63$$

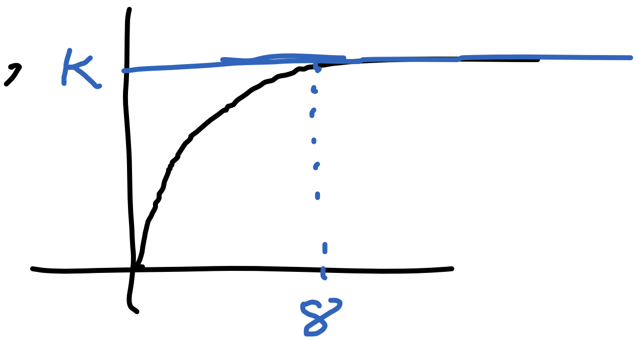
$$\text{at } t = 2T \Rightarrow c(t) = 0.86$$

$$t = 3T \Rightarrow c(t) = 0.95$$

$$t = 4T \Rightarrow c(t) = 0.98$$

$$R(s) \longrightarrow \boxed{\dfrac{K}{2s+1}} \longrightarrow$$

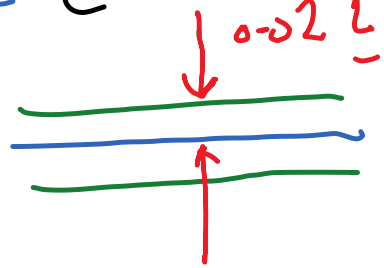
$T = 2 \,,\; K$



زمان نشست settling time : مدت زمان لازم برای اینکه پاسخ به حوالی مقدار نهایی برسد و از آن خارج نشود.

0.02 تا 0.05

$2\% \Rightarrow 1$



$\Rightarrow t_s = 4T \,,\; 2\%$

5%

$t_s = 3T \quad 5\%$

زمان صعود، Rise time t_r

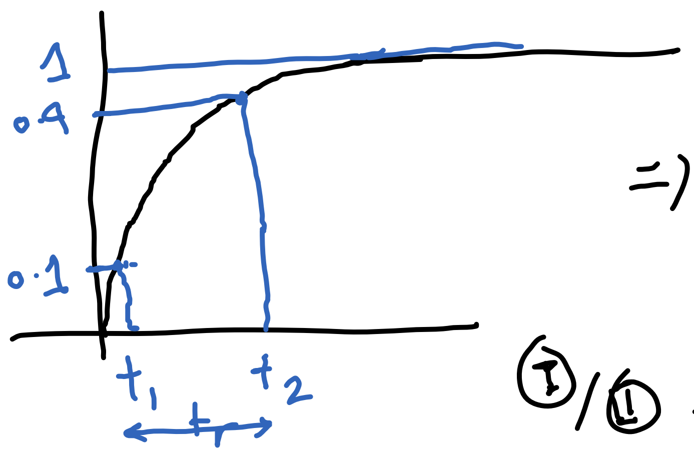
مدت زمان لازم برای اینکه پاسخ از حوالی مقدار اولیه به حوالی مقدار نهایی برسد.

۱۰۰٪ ، ۹۵٪ ، ۹۰٪

$0\% \Rightarrow t_r = \infty$

5% $\xrightarrow{\text{تمرین}}$ $t_r = ?$

$10\% \Rightarrow$ ✓ $\quad t_2 - t_1 = t_r$



$$c(t) = 1 - e^{-\frac{t}{T}}$$

$$\Rightarrow c(t_1) = 1 - e^{-\frac{t_1}{T}} = 0.1 \Rightarrow e^{-\frac{t_1}{T}} = 0.9 \quad (\mathrm{I})$$

$$c(t_2) = 1 - e^{-\frac{t_2}{T}} = 0.9 \Rightarrow e^{-\frac{t_2}{T}} = 0.1 \quad (\mathrm{II})$$

$$(\mathrm{I})/(\mathrm{II}) \Rightarrow e^{\frac{(t_2 - t_1)}{T}} = 9 \overset{Ln}{\Rightarrow} \frac{t_2 - t_1}{T} = \overset{2.2}{Ln 9} \Rightarrow \boxed{t_r = 2.2T}$$

پاسخ سیستم مرتبه اول به ورودی ضربه، پله، شیب

$$R(s) \longrightarrow \boxed{\frac{1}{Ts+1}} \longrightarrow C(s)$$

ضربه $R(s)=1 \Rightarrow C(s)=\dfrac{1}{Ts+1} \Rightarrow c(t)=\dfrac{1}{T}e^{-\frac{t}{T}}$　مشتق

$r(t)=\delta(t)$

پله $R(s)=\dfrac{1}{s} \Rightarrow C(s)=\dfrac{1}{Ts+1}\times\dfrac{1}{s}$　　$c(t)=1-e^{-\frac{t}{T}}$　مشتق

$r(t)=t$

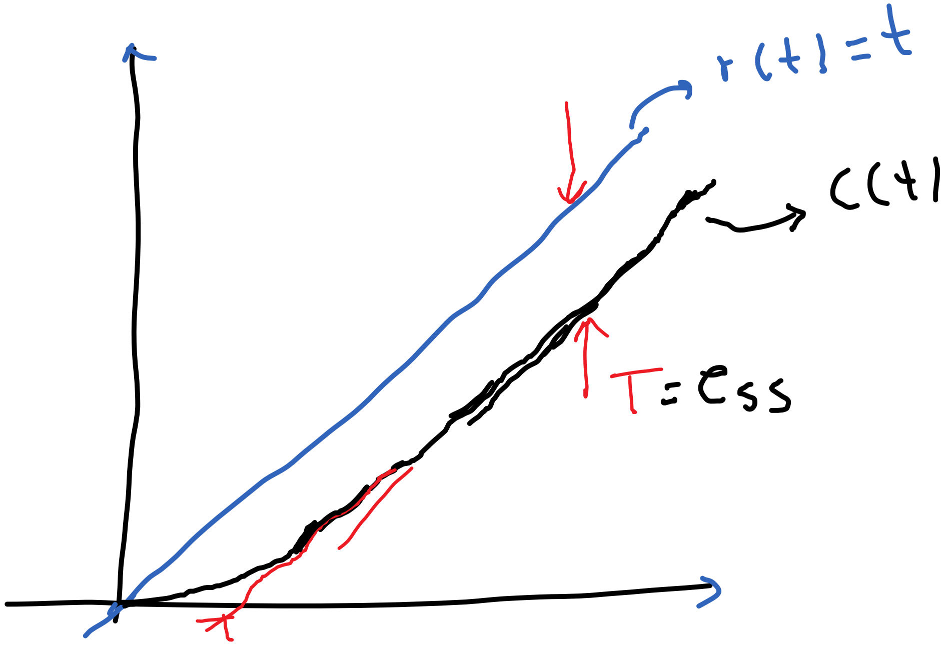
شیب $R(s)=\dfrac{1}{s^2} \Rightarrow C(s)=\dfrac{1}{Ts+1}\times\dfrac{1}{s^2}$ $\overset{\text{تمرین}}{\Rightarrow}$ $c(t)=-T+t+Te^{-\frac{t}{T}}$

$r(t)=\dfrac{t^2}{2}$

می‌توان پاسخ به مشتق یک سیگنال ورودی را با مشتق‌گیری از پاسخ سیستم به سیگنال اصلی به دست آورد.

$$r(t) \longrightarrow \boxed{G(s)} \longrightarrow c(t) \qquad\qquad \frac{d}{dt}r(t) \longrightarrow \boxed{G(s)} \longrightarrow \frac{d}{dt}c(t)$$

$$c(t) = -T + t + Te^{-\frac{t}{T}}$$



خروجی - ورودی = خطای پاسخ

$$e(t) = r(t) - c(t)$$

$$\Rightarrow e(t) = t - \left(-T + t + Te^{-\frac{t}{T}}\right)$$

$$= T - Te^{-\frac{t}{T}}$$

$$e_{ss} = \lim_{t \to \infty} e(t) = T$$

خطای حالت ماندگار

سیستم های مرتبه دوم

$$R(s) \longrightarrow \boxed{\dfrac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}} \longrightarrow C(s)$$

ζ : ضریب میرایی

ω_n : فرکانس طبیعی نامیرا

$$s^2+2\zeta\omega_n s+\omega_n^2=0 \Rightarrow P_{1,2}=-\zeta\omega_n \pm \sqrt{\zeta^2\omega_n^2-\omega_n^2}$$

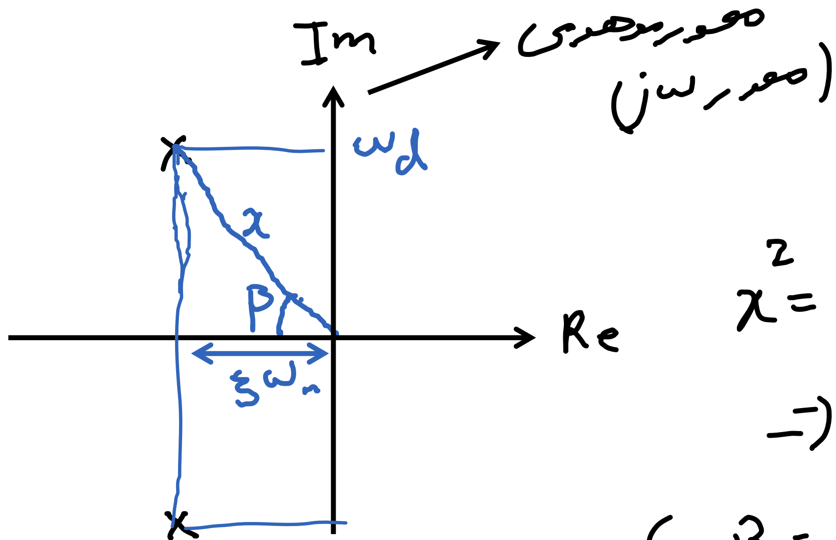
$$\Rightarrow P_{1,2}=-\zeta\omega_n \pm \omega_n\sqrt{\zeta^2-1}$$

حالت زیرمیرا $\quad 0<\xi<1$

$$p_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$$

$$\Rightarrow \quad p_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{1-\xi^2}\; j = -\xi\omega_n \pm \omega_d\, j$$

که $\quad \omega_d = \omega_n\sqrt{1-\xi^2}$



ω_d: فرکانس طبیعی میرا

$$x^2 = \xi^2\omega_n^2 + \omega_d^2 = \xi^2\omega_n^2 + \omega_n^2(1-\xi^2)$$

$$\Rightarrow \quad x = \omega_n$$

$$\cos\beta = \frac{\xi\omega_n}{\omega_n} = \xi \quad \Rightarrow \quad \beta = \cos^{-1}\xi$$

پاسخ سیستم مرتبه دوم به ورودی پله، حالت زیر میرا، $0<\xi<1$

$$R(s) \longrightarrow \boxed{G(s)} \longrightarrow C(s)$$

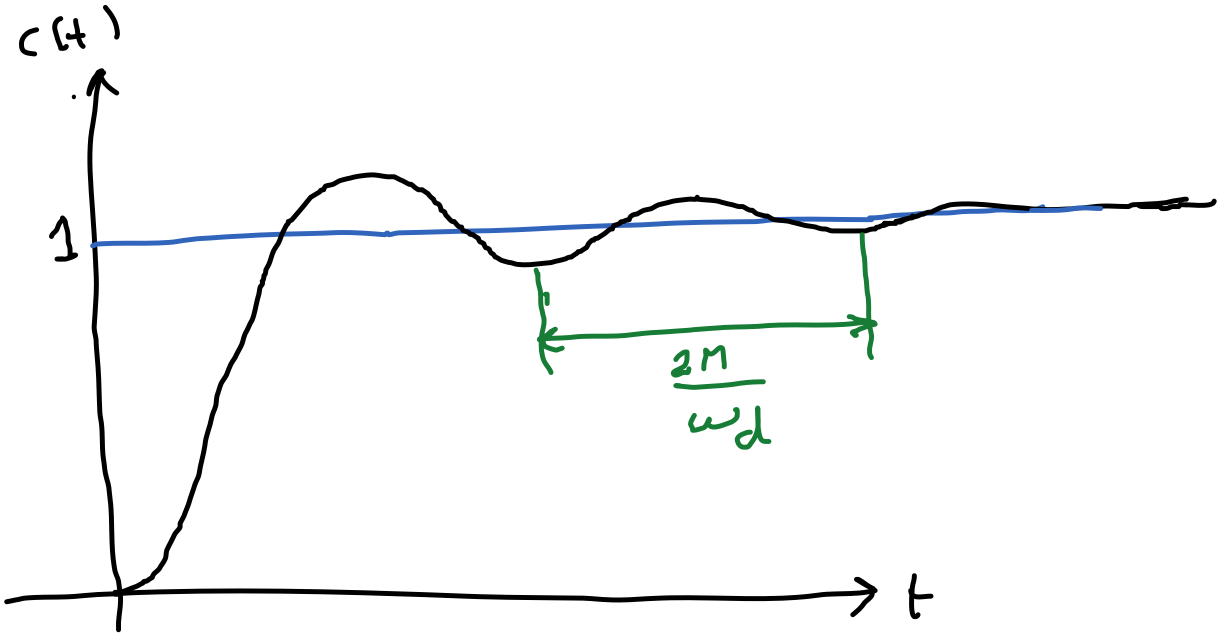
$$C(s)=\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}\times\frac{1}{s}=\frac{1}{s}-\frac{s+2\xi\omega_n}{s^2+2\xi\omega_n s+\omega_n^2}$$

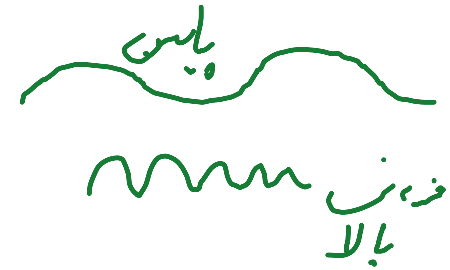
$$=\frac{1}{s}-\frac{s+\xi\omega_n+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_n^2-\xi^2\omega_n^2}=\frac{1}{s}-\frac{s+\xi\omega_n}{(s+\xi\omega_n)^2+\omega_d^2}-\frac{\frac{\xi\omega_n}{\omega_d}\,\omega_d}{(s+\xi\omega_n)^2+\omega_d^2}$$

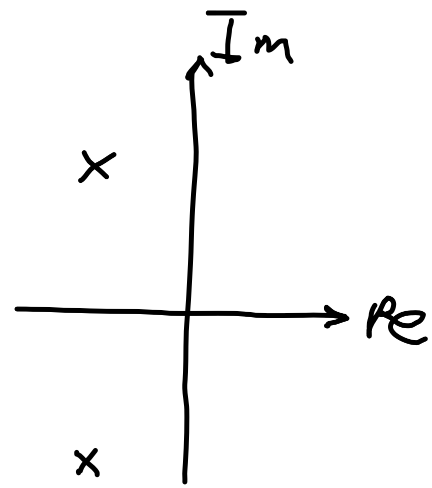
$$\Rightarrow c(t)=1-e^{-\xi\omega_n t}\left(\cos\omega_d t+\frac{\xi\omega_n}{\omega_d}\sin\omega_d t\right)=1-\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t+\beta)$$

تمرین

که $\beta=\cos^{-1}\xi$ ،

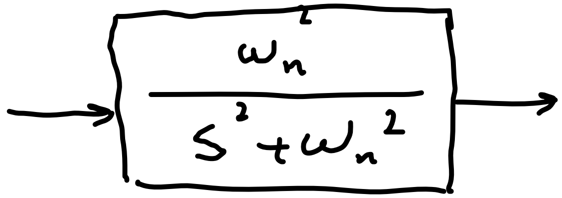




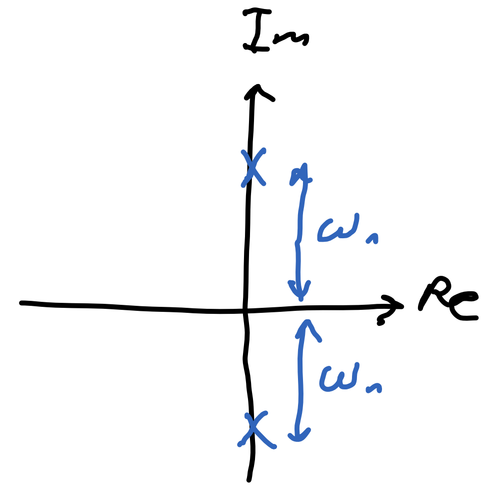


$$\omega_d = \omega_n \sqrt{1-\zeta^2}$$

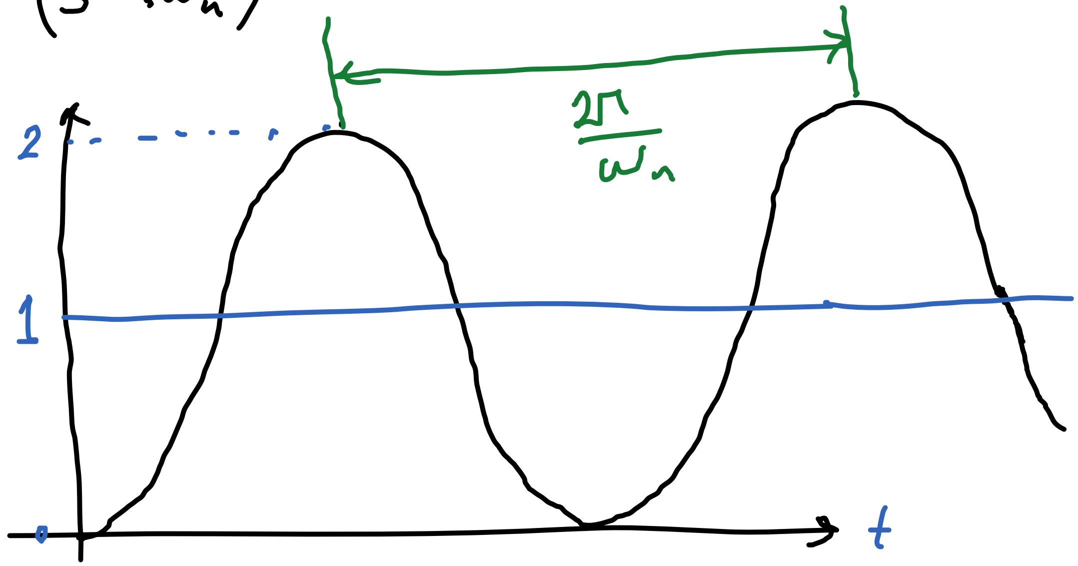
حالت نامیرا $\zeta = 0$



$$\Rightarrow P_{1,2} = \pm \omega_n j$$



$$C(s) = \frac{\omega_n^2}{(s^2+\omega_n^2)} \times \frac{1}{s} \Rightarrow c(t) = 1 - \cos \omega_n t$$



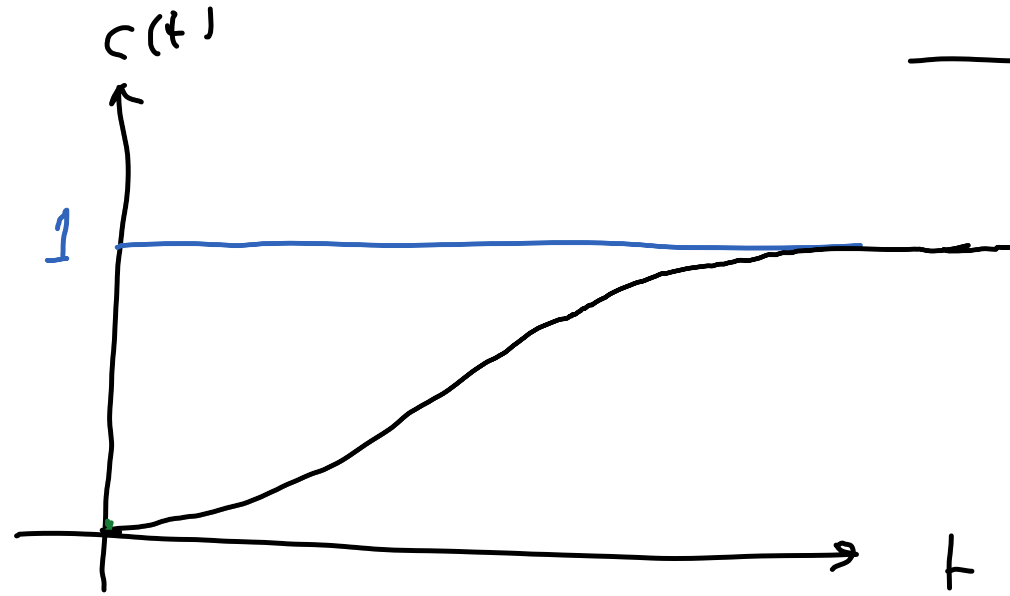
حالت میرایی بحرانی

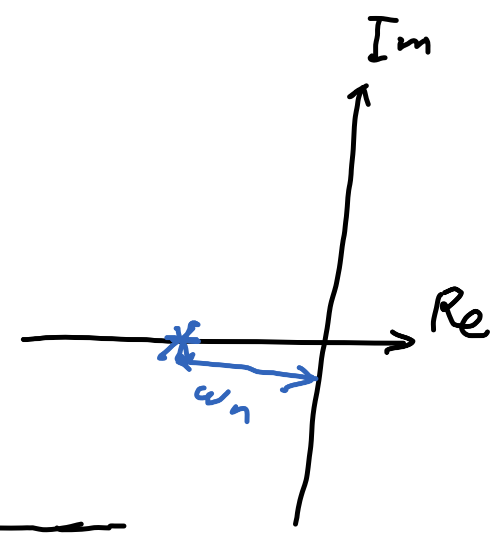
$$\zeta = 1$$

$$C(s) = \frac{\omega_n^2}{(s+\omega_n)^2} \times \frac{1}{s}$$

$$\Rightarrow \quad c(t) = 1 - e^{-\omega_n t}(1+\omega_n t)$$

$$P_{1,2} = \omega_n$$



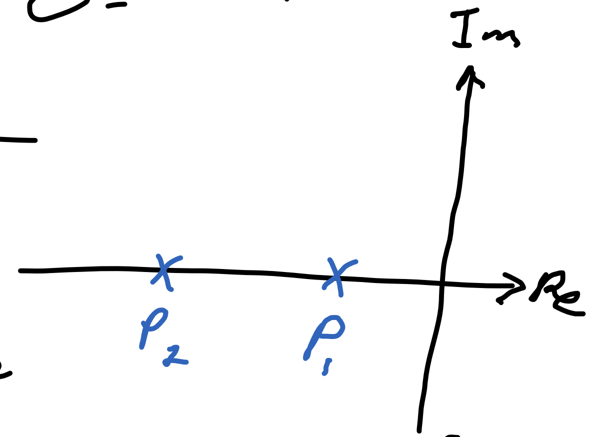


حالت فوق میرا $\xi > 1$

$$P_{1,2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$$

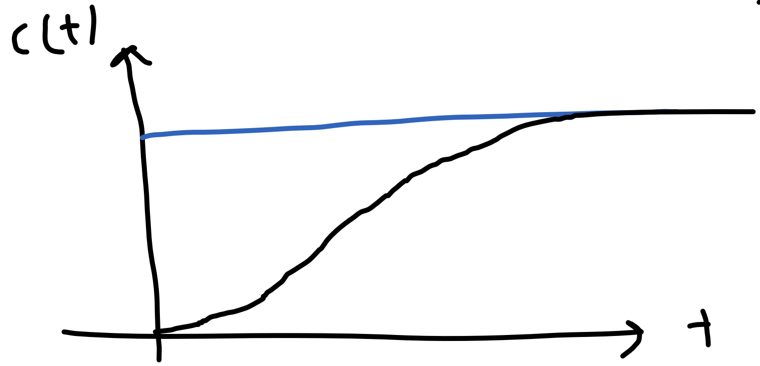
دو قطب حقیقی مجزا داریم

$$C(s) = \frac{\omega_n^2}{(s-P_1)(s-P_2)} \times \frac{1}{s} = \frac{1}{s} + \frac{a_1}{s-P_1} + \frac{a_2}{s-P_2}$$



$$c(t) = 1 + a_1 e^{P_1 t} + a_2 e^{P_2 t}$$

$$\Rightarrow a_1 = \frac{\omega_n^2}{P_1(P_1-P_2)}, \quad a_2 = \frac{\omega_n^2}{P_2(P_2-P_1)}$$



$$\Rightarrow c(t) = 1 + \frac{P_1P_2}{P_1-P_2}\left(\frac{e^{P_1 t}}{P_1} - \frac{e^{P_2 t}}{P_2}\right)$$

$$\Rightarrow c(t) = 1 + \frac{\omega_n^2}{2\omega_n\sqrt{1-\xi^2}}\left(\frac{e^{P_1 t}}{P_1} - \frac{e^{P_2 t}}{P_2}\right)$$

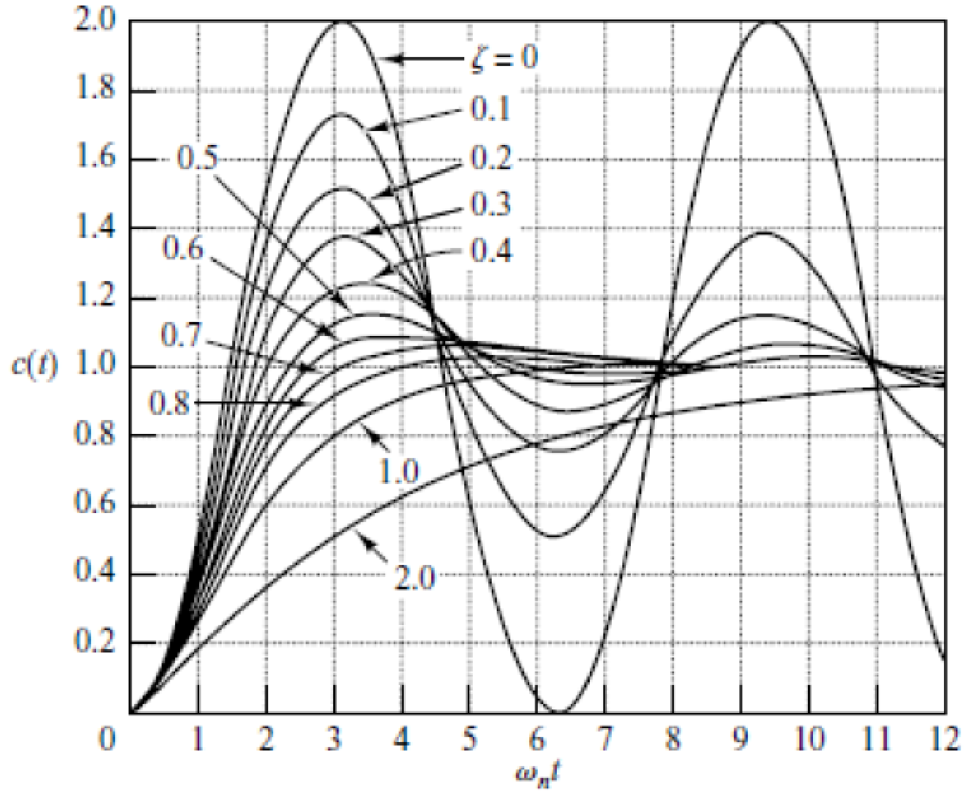
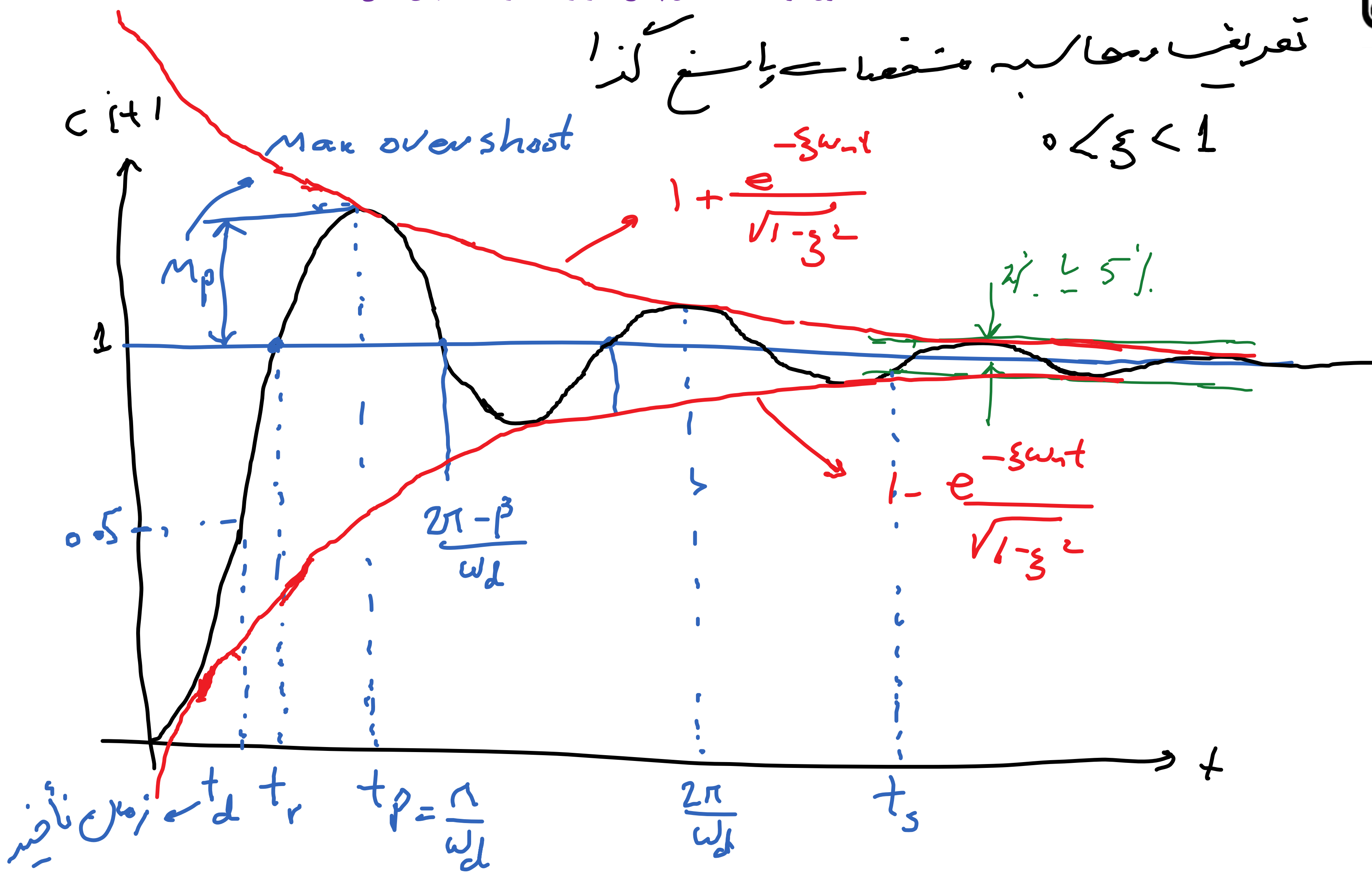


Figure 1: Unit step response curves of the system

تعریف و محاسبه مشخصات پاسخ گذرا

$0 < \xi < 1$



زمان اوج، Peak time t_p

$$c(t)=1-e^{-\xi\omega_n t}\left(\cos\omega_d t+\frac{\xi\omega_n}{\omega_d}\sin(\omega_d t)\right)$$

$$\frac{dc(t)}{dt}=0 \Rightarrow \dots \text{(تمرین)} \Rightarrow \frac{e^{-\xi\omega_n t}\,\omega_n}{\sqrt{1-\xi^2}}\sin\omega_d t=0$$

$$\Rightarrow \sin\omega_d t=0 \Rightarrow \omega_d t=0,\underbrace{\pi},2\pi,\dots \Rightarrow \boxed{t_p=\frac{\pi}{\omega_d}}$$

ماکزیمم فراجهش، Mp

$$M_p=c(t_p)-1$$

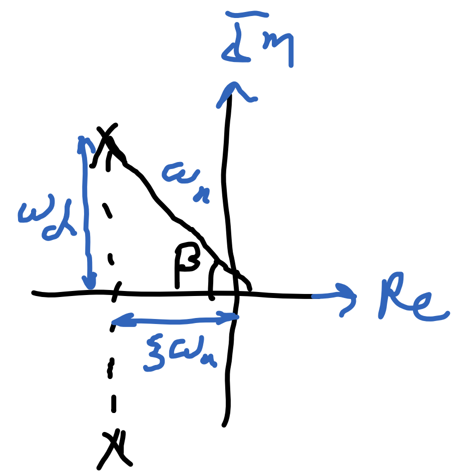
$$M_p = 1 - \frac{e^{-\xi\omega_n t_p}}{\sqrt{1-\xi^2}}\sin(\omega_d t_p+\beta) - 1 =$$

$\sin(\pi+\beta) = -\sin\beta$

$$\Rightarrow M_p = -\frac{e^{-\xi\omega_n \frac{\pi}{\omega_d}}}{\sqrt{1-\xi^2}}\sin\left(\omega_d \frac{\pi}{\omega_d}+\beta\right)$$

$$M_p = \frac{e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}}}{\cancel{\sqrt{1-\xi^2}}}\cancel{\sin\beta} \quad , \quad \sin(\cos^{-1}\xi)$$

$$\sin\beta = \frac{\omega_d}{\omega_n} = \sqrt{1-\xi^2}$$



$$\boxed{M_p = e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}}}$$

t_r زمان اوج

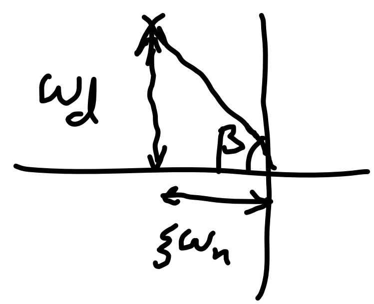
$$c(t_r) = 1$$

$$-e^{-\xi\omega_n t}\left(\cos \omega_d t + \frac{\xi\omega_n}{\omega_d}\sin(\omega_d t)\right) = 0$$

$$\cos(\omega_d t) + \frac{\xi}{\sqrt{1-\xi^2}}\sin \omega_d t = 0 \Rightarrow \frac{\xi}{\sqrt{1-\xi^2}}\sin \omega_d t = -\cos \omega_d t$$

$$\Rightarrow \frac{\xi}{\sqrt{1-\xi^2}}\,\mathrm{tg}\,\omega_d t = -1 \Rightarrow \mathrm{tg}\,\omega_d t_r = -\frac{\sqrt{1-\xi^2}}{\xi} = \frac{-\omega_d}{\xi\omega_n} = -\mathrm{tg}\,\beta$$

$$\mathrm{tg}\,\beta = \frac{\omega_d}{\xi\omega_n}$$



$$\Rightarrow \omega_d t_r = \pi - \beta$$

$$\Rightarrow t_r = \frac{\pi-\beta}{\omega_d}$$

$$c(t_r) = 1 \Rightarrow \sin(\omega_d t_r + \beta) = 0$$

$$\omega_d t_r + \beta = 0, \pi, 2\pi, \ldots \Rightarrow \boxed{t_r = \frac{\pi - \beta}{\omega_d}}$$

t_s زمان نشست

$$\boxed{t_s \simeq \begin{cases} \dfrac{4}{\xi \omega_n} & 2\% \\[2ex] \dfrac{3}{\xi \omega_n} & 5\% \end{cases}}$$

Impulse Response

پاسخ به ورودی ضربه برای سیستم مرتبه دوم:

$$R(s)=1 \Rightarrow C(s)=\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

Step Res.

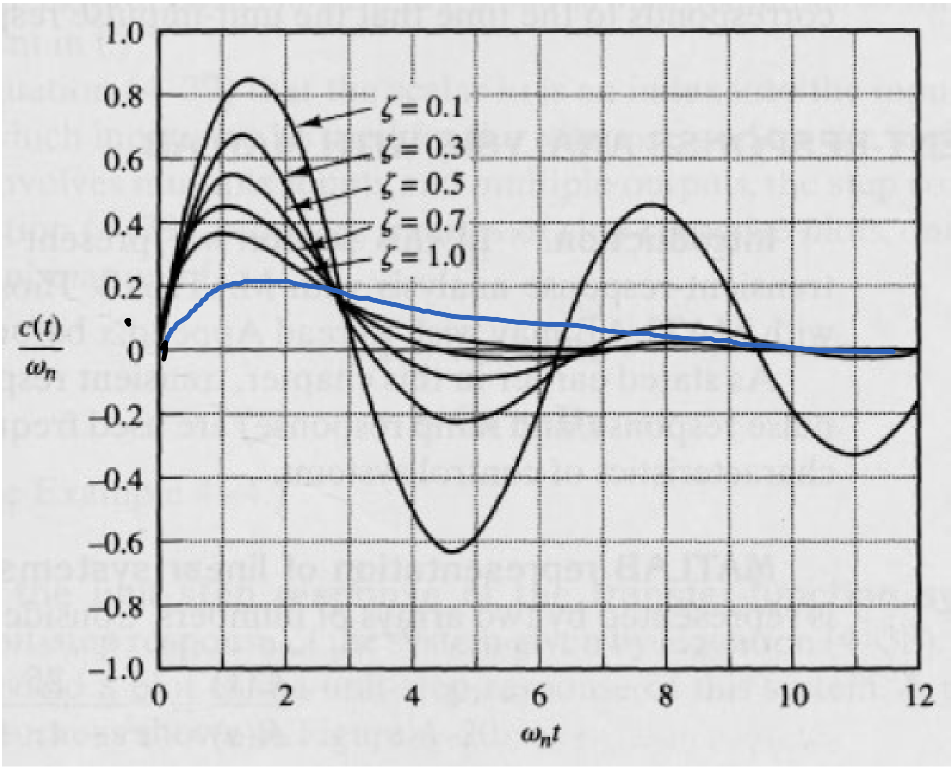
$$\xi<1 \Rightarrow 1-\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t+\beta) \Rightarrow c(t)=\frac{\omega_n}{\sqrt{1-\xi^2}}e^{-\xi\omega_n t}\sin\omega_d t$$

$$\xi=0 \Rightarrow c(t)=1-\cos\omega_n t \Rightarrow c(t)=\omega_n\sin\omega_n t$$

$$\xi=1 \Rightarrow c(t)=1-e^{-\omega_n t}(1+\omega_n t) \Rightarrow c(t)=\omega_n^2 t\, e^{-\omega_n t}$$

$$\xi>1 \Rightarrow c(t)=1+\frac{\omega_n}{2\sqrt{1-\xi^2}}\left(\frac{e^{p_1 t}}{p_1}-\frac{e^{p_2 t}}{p_2}\right) \Rightarrow c(t)=\frac{\omega_n}{2\sqrt{1-\xi^2}}\left(e^{p_1 t}-e^{p_2 t}\right)$$

Impulse Response



Unit ramp response of a second order system

$r(t)=t$

$$C(s)=\frac{\omega_n^{\ 2}}{s^2+2\zeta\omega_n+\omega_n^{\ 2}}\cdot\frac{1}{s^2} \rightarrow \frac{a_1}{s}+\frac{a_2}{s^2}+\frac{bs+c}{s^2+2\zeta\ -}$$

R(s) = 1/ s^2

for an underdamped system $(0<\zeta<1)$

$$c(t)=t-\frac{2\zeta}{\omega_n}+e^{-\zeta\omega_n t}\left(\frac{2\zeta}{\omega_n}\cos\omega_d t+\frac{2\zeta^2-1}{\omega_n\sqrt{1-\zeta^2}}\sin\omega_d t\right)\qquad t\geq 0$$

and the error:

$$e(t)=r(t)-c(t)=t-c(t)$$

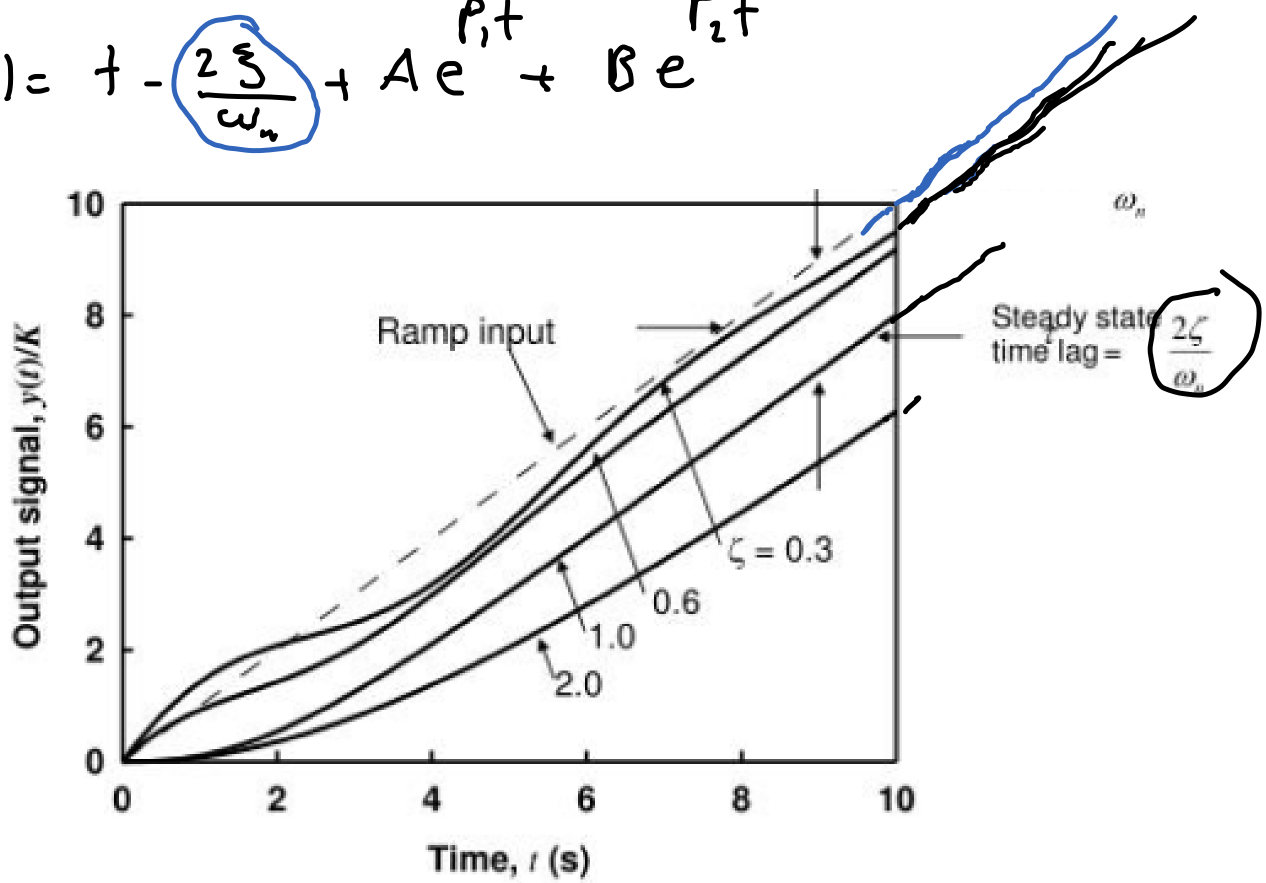
at steady-state:

$$e(\infty)=\lim_{t\to\infty}e(t)=\frac{2\zeta}{\omega_n}$$

Ramp Response

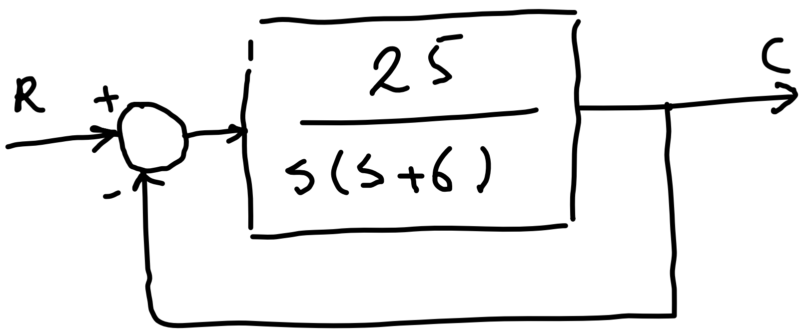
$$\xi = 1 \Rightarrow c(t) = t - \frac{2}{\omega_n} + \frac{2}{\omega_n} e^{-\omega_n t}\left(1 + \frac{\omega_n t}{2}\right)$$

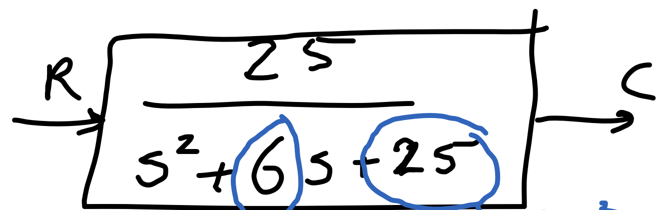
$$\xi > 1 \Rightarrow c(t) = t - \frac{2\xi}{\omega_n} + A e^{p_1 t} + B e^{p_2 t}$$



به ورودی پله

مثال: برای سیستم زیر، t_s ، t_p ، M_p و t_r را محاسبه کنید و شکل تقریبی جواب را رسم کنید.





$$\frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

$\omega_n^2 \longrightarrow \omega_n=5$

$2\xi\omega_n=6 \Rightarrow \xi=\frac{6}{10}=0.6$

$$t_s=\frac{4}{\xi\omega_n}=\frac{4}{0.6\times 5}=\frac{4}{3}=1.33s \quad \underline{2\%} \quad \checkmark$$

$$t_s=\frac{3}{\xi\omega_n}=\frac{3}{3}=1s \quad 5\%$$

$$t_p=\frac{\pi}{\omega_d}=\frac{\pi}{\omega_n\sqrt{1-\xi^2}}=\frac{3.14}{4}$$

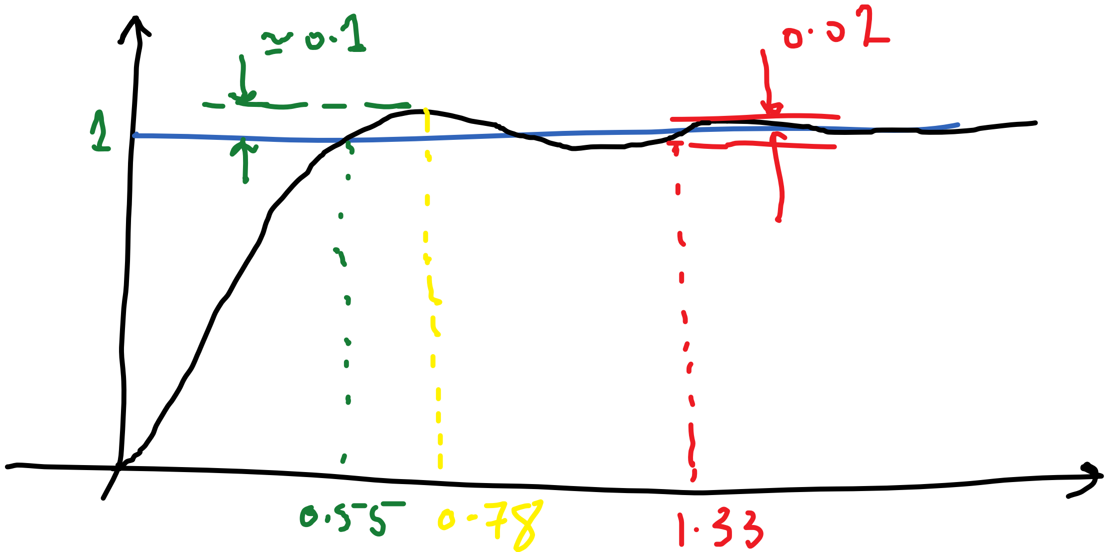
$$\Rightarrow t_p=0.78s$$

$$t_r = \frac{\pi - \beta}{\omega_d} = \frac{3.14 - 0.43}{4} = 0.55\ s \quad , \quad t_p = 0.78$$

$$t_s = 1.33$$

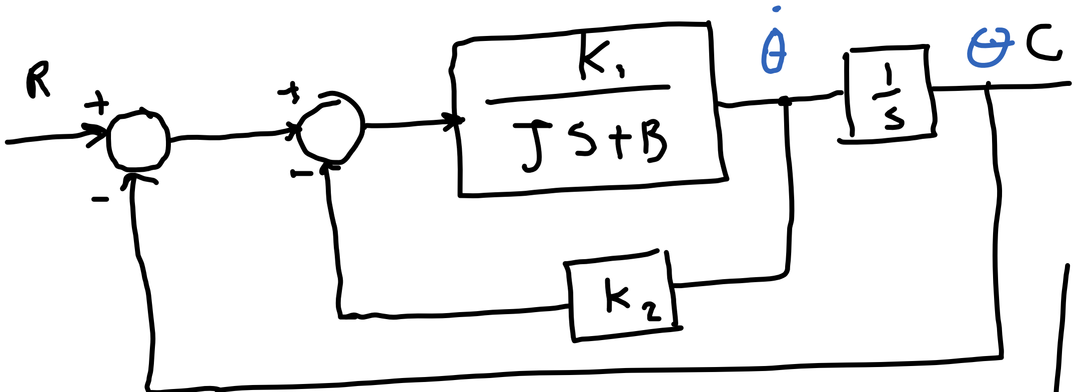
$$\beta = \cos^{-1}\zeta = \cos^{-1}0.6 = 0.43\ rad$$

$$M_p = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} = e^{\frac{-3\pi}{4}} = 0.095 \times 100 = 9.5\%$$



مثال: برای سیستم موتور سرو زیر مقادیر K_1 و K_2 را به گونه‌ای بدست آورید که

$Mp = 20\%$ و $t_p = 1s$



$$\frac{C}{R} = \frac{K_1}{Js^2 + (B + K_1K_2)s + K_1}$$

$$\frac{C}{R} = \frac{K_1/J}{s^2 + \underbrace{\left(\frac{B + K_1K_2}{J}\right)}_{2\zeta\omega_n}s + \underbrace{\frac{K_1}{J}}_{\omega_n^2}}$$

$$Mp = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} = 0.2$$

$$\overset{Ln}{\Longrightarrow} \frac{-\zeta\pi}{\sqrt{1-\zeta^2}} = Ln\, 0.2 = -1.6$$

$$\Rightarrow \frac{\zeta}{\sqrt{1-\zeta^2}} = \frac{1.6}{\pi} \Rightarrow \frac{\zeta^2}{1-\zeta^2} = (0.51)^2 = 0.26 \Rightarrow (1-\zeta^2)0.26 = \zeta^2$$

$$(1-\xi^2)\,0.26 = \xi^2 \Rightarrow 1.26\,\xi^2 = 0.26 \Rightarrow \xi^2 = \frac{0.26}{1.26} = 0.2$$

$$\Rightarrow \boxed{\xi = 0.45}$$

در صورت سوال داریم

$$t_p = \frac{\pi}{\omega_d} = \frac{3.14}{\omega_n\sqrt{1-\xi^2}} = 1 \Rightarrow \boxed{\omega_n = 3.53} \quad \text{و} \quad \underline{B = 1 \text{ و } J = 1}$$

$$\frac{K_1}{J} = \omega_n^2 \Rightarrow K_1 = 3.53^2 = \boxed{12.5}$$

$$\frac{B + K_1 K_2}{J} = 2\xi\omega_n \Rightarrow \frac{1 + 12.5\,K_2}{1} = 2 \times 0.45 \times 3.53 \Rightarrow \boxed{K_2 = 0.18}$$

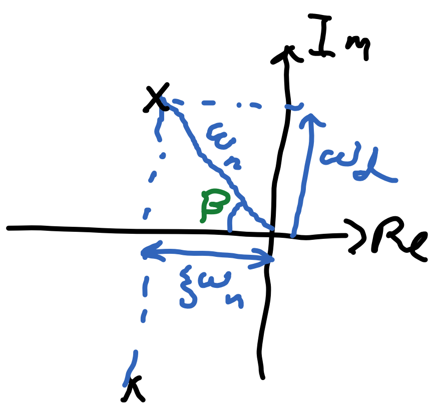
$$K_1 \uparrow \Rightarrow \omega_n \uparrow$$

$$K_2 \uparrow \Rightarrow \xi \uparrow \rightarrow M_p \downarrow$$

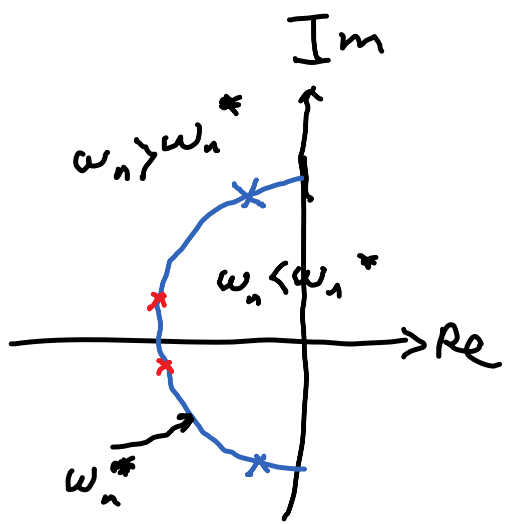
برای سیستم های مرتبه ۲، مکان هندسی پارامترهای ثابت

$$\frac{C}{R} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

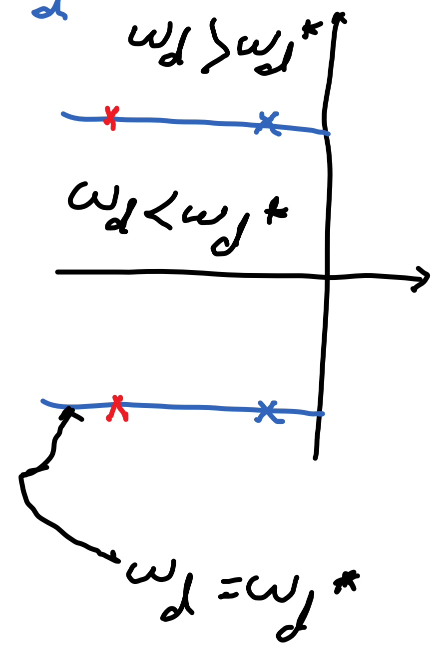
$$P_{1,2} = -\xi\omega_n \pm \underbrace{\omega_n\sqrt{1-\xi^2}}_{\omega_d}\, j \qquad \xi < 1$$

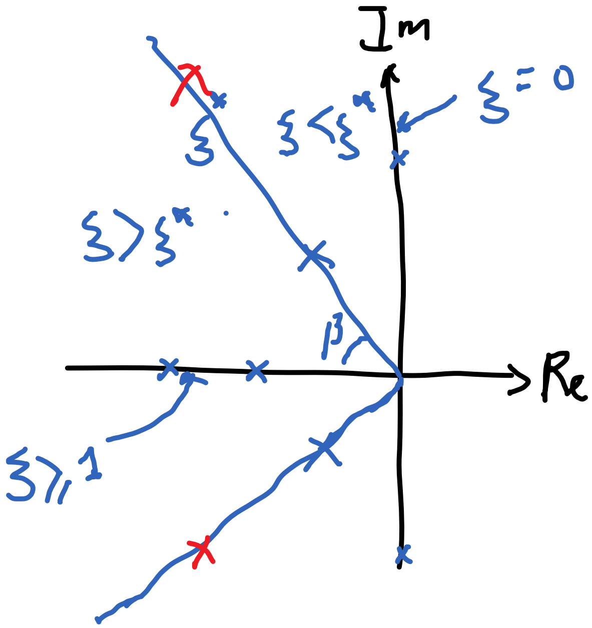


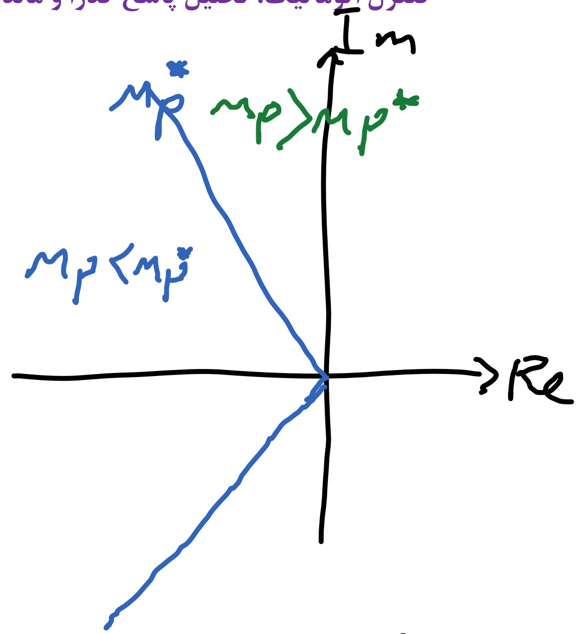
$$\beta = \cos^{-1}\xi$$

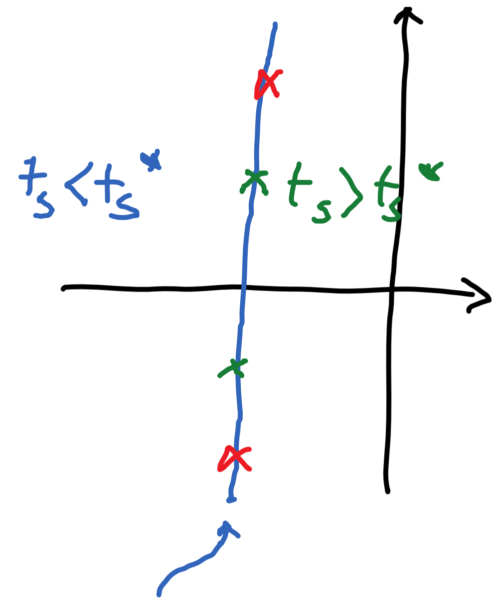


$$\omega_n = \omega_n^*$$









$\xi = \xi^*$

$\beta = \cos^{-1}\xi$

$\cos\beta = \xi$

$M_p = M_p^*$

$M_p = e^{\frac{-\xi\pi}{\sqrt{1-\xi^2}}}$

$t_s = t_s^*$

$t_s = \frac{4}{\xi\omega_n}$

مثال: برای یک سیستم مرتبه دوم محل قطبها را به نحوی تعیین کنید که مشخصات زیر برقرار باشد،

$0.1 < M_p < 0.2$ ، $1 < t_s < 2$ ، $3 < \omega_n < 4$

$2 < \omega_d < 3$

$$t_s = \frac{4}{\zeta\omega_n} = 2 \Rightarrow \zeta\omega_n = 2$$

$$t_s = \frac{4}{\zeta\omega_n} = 1 \Rightarrow \zeta\omega_n = 4$$

$$M_p = 0.1 \Rightarrow e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} = 0.1 \overset{\dots}{\Rightarrow} \zeta = 0.6 \Rightarrow \beta = \cos^{-1}\zeta = 53^\circ$$

$$M_p = 0.2 \Rightarrow \quad 〃 \quad = 0.2 \overset{\dots}{\Rightarrow} \zeta = 0.45 \Rightarrow \beta = \cos^{-1}\zeta = 62^\circ$$

$0.1 < M_p < 0.2$

$3 < \omega_n < \cancel{4}\ 5$ $\qquad 1 < t_s < 2$

$3\ \cancel{4} < \omega_d < \cancel{6}\ 5$

$t_s = 2 \Rightarrow \zeta\omega_n = 2$

$t_s = \dfrac{4}{\zeta\omega_n} = 1 \Rightarrow \zeta\omega_n = 4$

$M_p = 0.1 \Rightarrow e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} = 0.1 \Rightarrow \zeta = 0.6 \Rightarrow \beta = \cos^{-1}\zeta = 53^\circ$

$M_p = 0.2 \Rightarrow \sim = 0.2 \Rightarrow \zeta = 0.45 \Rightarrow \beta = \cos^{-1}\zeta = 62^\circ$