ریاضی مهندسی پیشرفته
معادلات مشتق جزئی

مسائل انتشار موج و ارتعاش

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جلسه بیست و پنجم

PDE

- مسائل انتقال حرارت

  $$\frac{\partial u}{\partial t} = \nabla^2 u$$

  - مختصات دکارتی
    - یک بعدی
    - دو بعدی
    - سه بعدی
  - مختصات استوانه‌ای
  - مختصات کروی

- مسائل انتشار موج

  $$\frac{\partial^2 u}{\partial t^2} = \nabla^2 u$$

  - مختصات دکارتی
    - یک بعدی
    - دو بعدی
    - سه بعدی
  - مختصات استوانه‌ای

روش حل

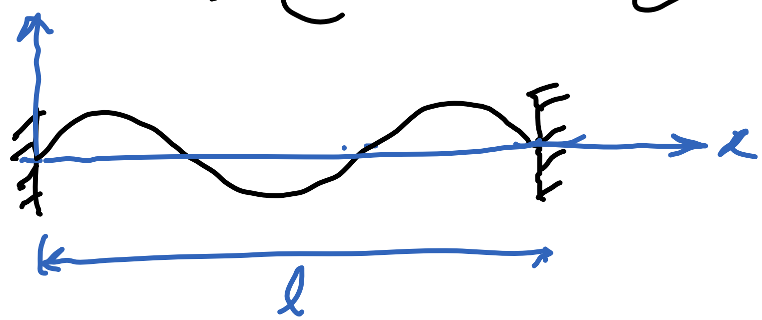
- روش جداسازی متغیرها ← همگن، $u(x,t) = \varphi(x)q(t)$
- استفاده از تبدیلات فوریه

تقسیم بندی از نظر شرایط مرزی

- محدود $0<x<L$ ، $-L<x<L$ ✓
- محدود، نیم محدود $0<x<\infty$ ✓
- نامحدود، نامحدود $-\infty<x<\infty$ ✓

- معادله
  - همگن ✓
  - ناهمگن ✓
- شرایط مرزی
- شرایط اولیه
  - همگن، ✓
  - ناهمگن ✓

مثال : انتشار موج در تار مرتعش



$$y_{tt}=c^2 y_{xx}, \quad 0<x<l, \quad t>0$$

BC: $y(0,t)=y(l,t)=0 \rightarrow X(0)=X(l)=0$

IC: $y(x,0)=f(x), \quad y_t(x,0)=g(x)$

روش جداسازی

$$y(x,t)=X(x)\,T(t)$$

$$\ddot{T}X=c^2TX'' \xrightarrow{\frac{1}{XT}} \frac{\ddot{T}}{c^2T}=\frac{X''}{X}=-\lambda^2=k$$

$$X''+\lambda^2X=0 \Rightarrow X_n(x)=\sin\lambda_n x, \quad \lambda_n=\frac{n\pi}{L}$$

$X(0)=X(l)=0$

اگر $\lambda^2=k$ به جواب بدیهی می‌رسیم

درحقیقت λ_n فرکانس‌های طبیعی و X_n مودهای ارتعاشی تار هستند.

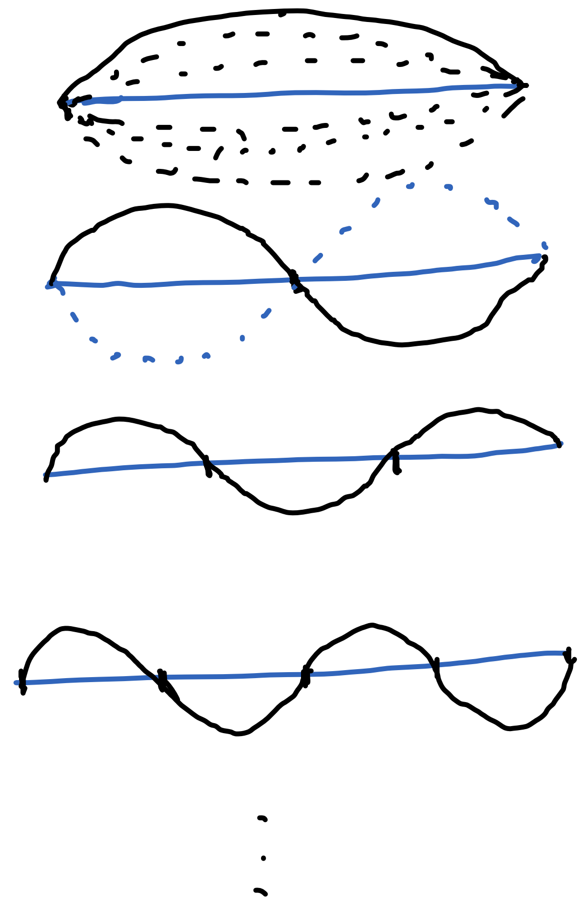
$$n=1, \quad \lambda_1=\frac{\pi}{L}, \quad X_1(x)=\sin\frac{\pi x}{L}$$

$$\lambda_2=\frac{2\pi}{L}, \quad X_2(x)=\sin\frac{2\pi x}{L}$$

$$\lambda_3=\frac{3\pi}{L}, \quad X_3(x)=\sin\frac{3\pi x}{L}$$

$$\lambda_4=\frac{4\pi}{L}, \quad X_4(x)=\sin\frac{4\pi x}{L}$$

$$\vdots$$



پاسخ زمانی

$$\frac{\ddot{T}}{c^2 T} = -\lambda_n^2 \Rightarrow \ddot{T} + c^2\lambda_n^2 T = 0$$

روی $T(0)$ شرط خاصی نداریم

$$\Rightarrow T(t) = A_n \cos c\lambda_n t + B_n \sin c\lambda_n t$$

با استفاده از اصل برهم نهی پاسخ عمومی به صورت زیر خواهد شد

$$y(x,t) = \sum_{n=1}^{\infty} X_n(x)\, T_n(t) = \sum_{n=1}^{\infty} \sin(\lambda_n x)\left(A_n \cos(c\lambda_n t) + B_n \sin(c\lambda_n t)\right)$$

در این پاسخ $\lambda_n = \frac{n\pi}{L}$ را بسامد مکانی و $c\lambda_n = \frac{n\pi c}{L}$ را بسامد زمانی می گویند

$$y(x,t) = \sum_{n=1}^{\infty} \sin(\lambda_n x)\left(A_n \cos(c\lambda_n t) + B_n \sin(c\lambda_n t)\right)$$

$$y_t(x,t) = \sum_{n=1}^{\infty} \sin \lambda_n x \left(-A_n c\lambda_n \sin(c\lambda_n t) + B_n c\lambda_n \cos(c\lambda_n t)\right)$$

اعمال IC

$$y(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin(\lambda_n x) \Rightarrow A_n = \frac{2}{l}\int_0^l f(x)\sin(\lambda_n x)\,dx$$

$$y_t(x,0) = g(x) = \sum_{n=1}^{\infty} c\lambda_n B_n \sin(\lambda_n x) \Rightarrow c\lambda_n B_n = \frac{2}{l}\int_0^l g(x)\sin(\lambda_n x)\,dx$$

حال فرض کنید $g(x)=0$ باشد پس $B_n=0$ است.

$$y(x,t) = \sum_{n=1}^{\infty} A_n \sin(\lambda_n x)\cos(c\lambda_n t)$$

$$y(x,t)=\sum_{n=1}^{\infty} A_n \sin(\lambda_n x)\cos(c\lambda_n t)$$

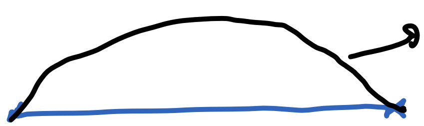
از رابطه مثلثاتی داریم

$$y(x,t)=\frac{1}{2}\sum_{n=1}^{\infty} A_n \left\{\sin\lambda_n(x-ct)+\sin\lambda_n(x+ct)\right\}$$

دانستیم که $f(x)=\sum_{n=1}^{\infty} A_n \sin\lambda_n x \Rightarrow f(x-ct)=\sum_{n=1}^{\infty} A_n \sin\lambda_n (x-ct)$

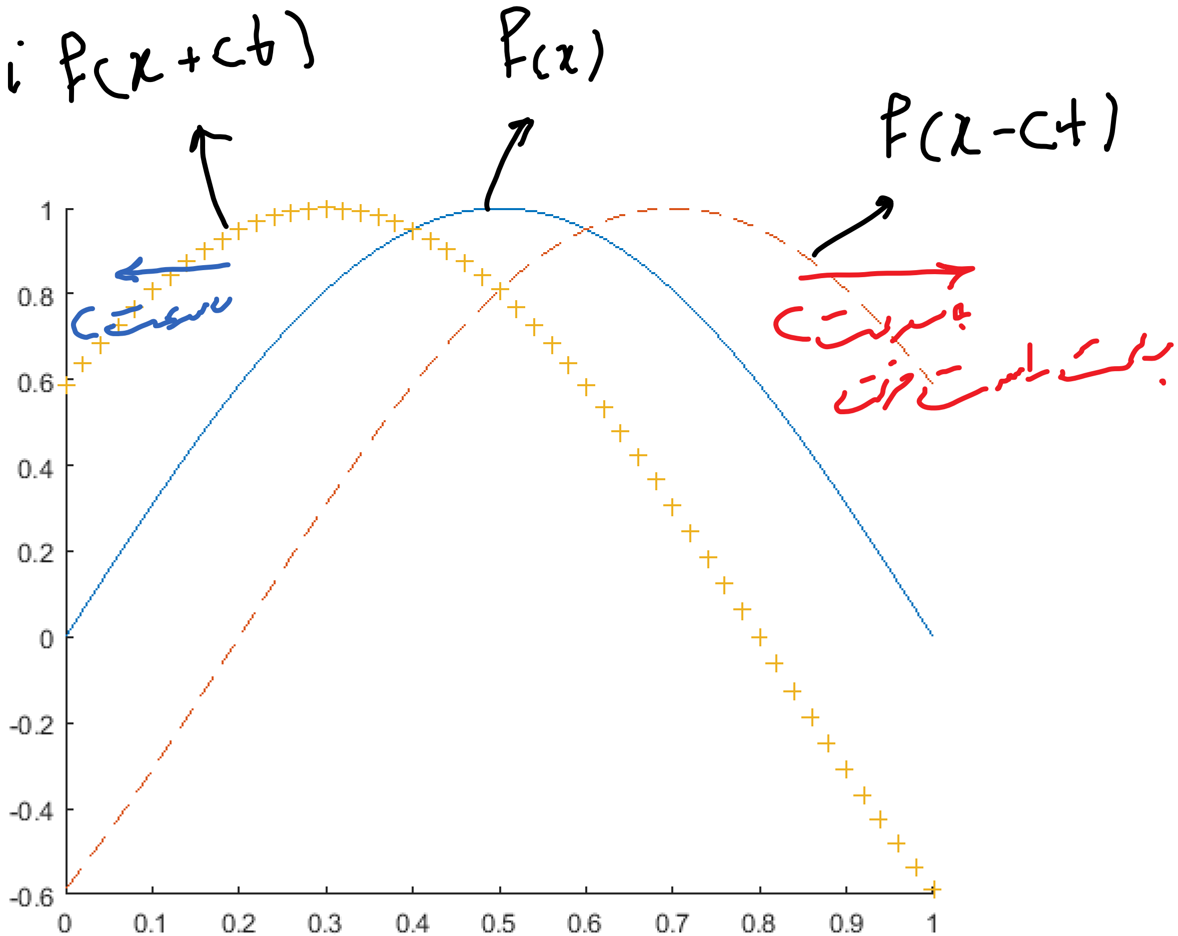
$$\Rightarrow y(x,t)=\frac{1}{2}\left\{f(x-ct)+f(x+ct)\right\}$$

با فرض اینکه $f(x)=\sin\frac{\pi x}{L}$ پاسخ را به دست آورید

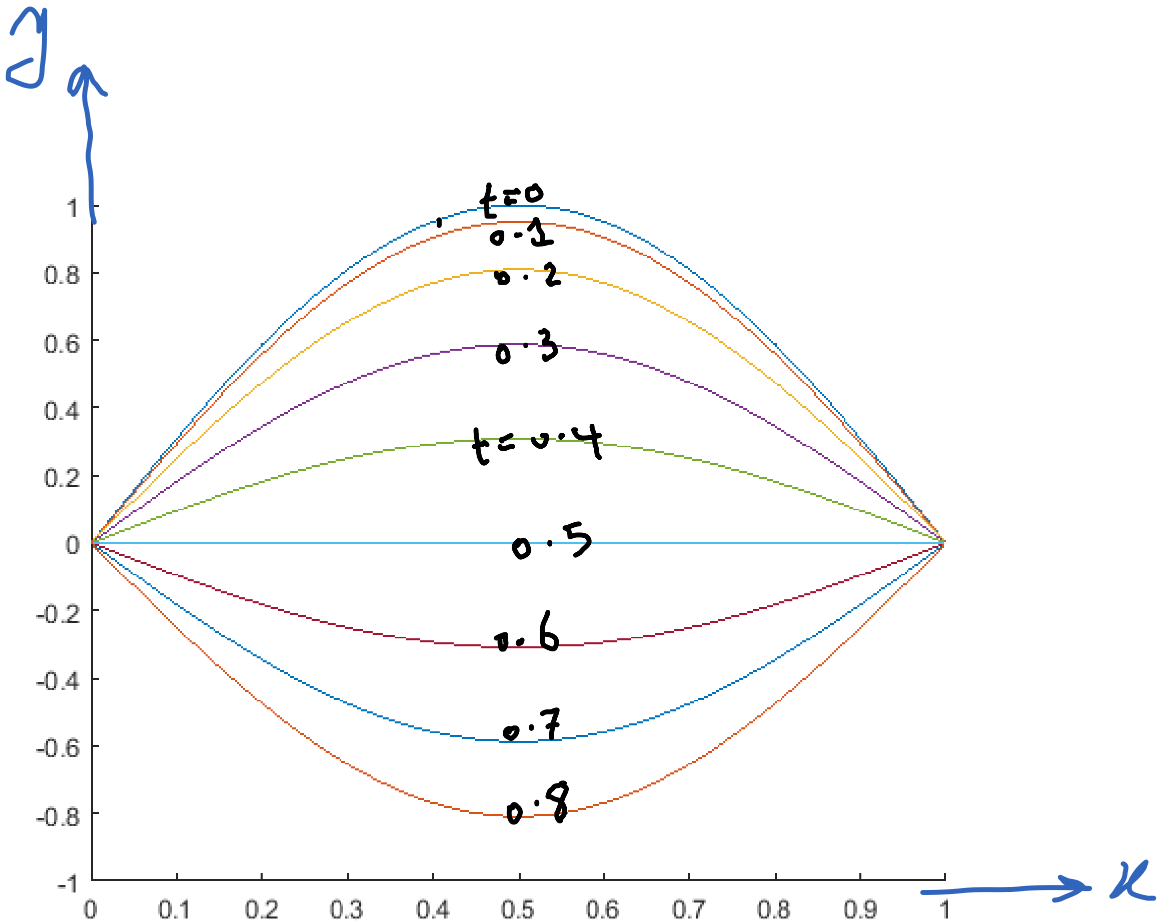


شکل اولیه $f(x)$

f(x+ct) یا f(x) که با سرعت c به
سمت چپ حرکت می‌کند




```
clc
clear
L=1;
c=1;
x=0:0.02:L;
%%%%%%%%
t=.2;
f=sin(pi*x/L);
fcm=sin(pi*(x-c*t)/L);
fcp=sin(pi*(x+c*t)/L);
figure (1)
hold on
plot(x,f)
plot(x,fcm,'--')
plot(x,fcp,'+')
```

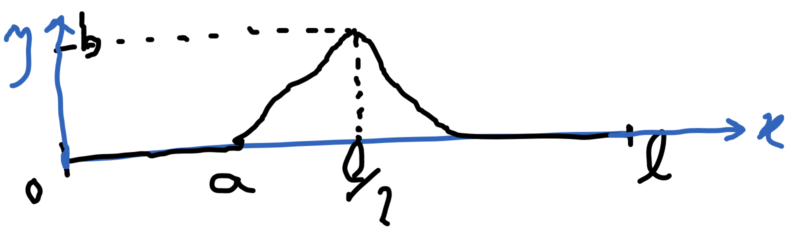



```
clc
clear
L=1;
c=1;
x=0:0.02:L;
t=0;
for i=1:200
fcm=sin(pi*(x-c*t)/L);
fcp=sin(pi*(x+c*t)/L);
y=0.5*(fcp+fcm);
figure (1)
plot(x,y)
xlim([0 L])
ylim([-1 1])
pause(.1)
t=t+0.02;
end
```

چون $f(x)$ بصورت $\sin\frac{\pi x}{L}$ بود، بنابراین فقط مود اول تحریک می‌شود و عملاً مودهای بعدی را در نوسانات تار نمی‌بینیم

شبیه سازی دیگر

$$y(x,0)=f(x)$$



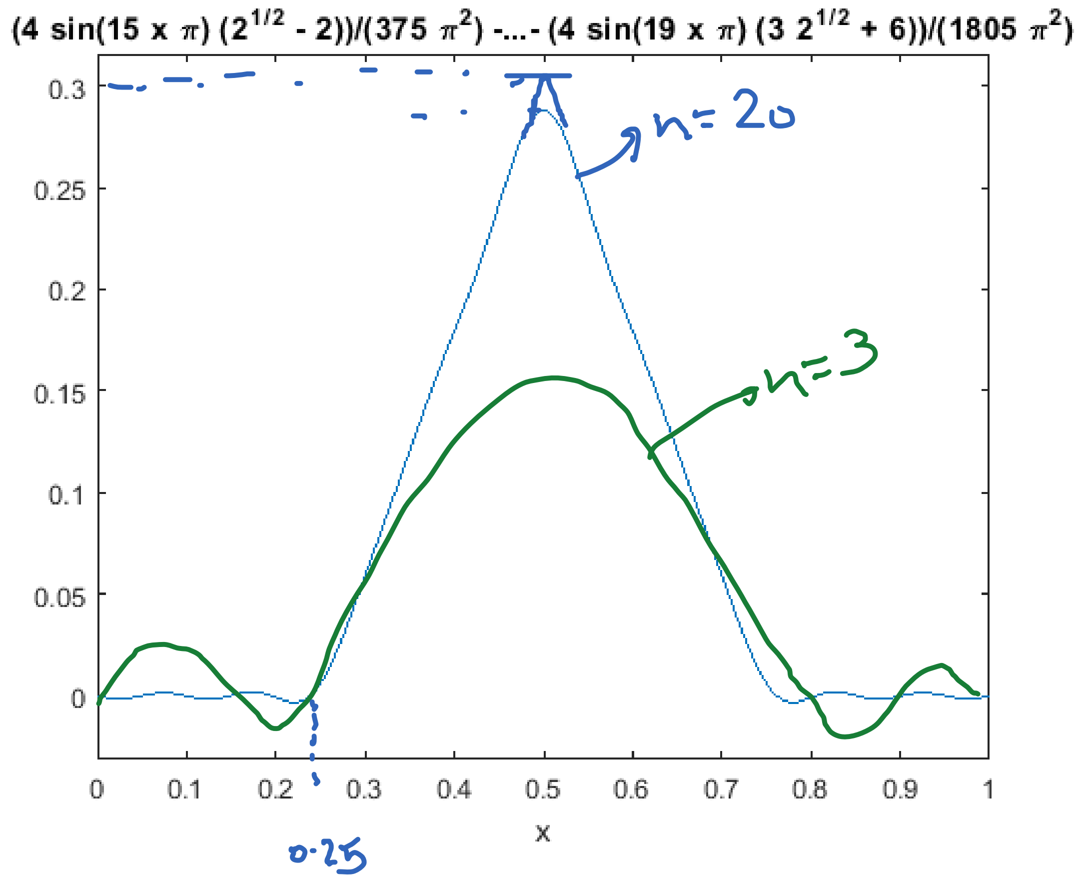
$$f(x)=\begin{cases} 0 & x<a \\ \dfrac{b}{a}(x-a) & a<x<\dfrac{l}{2} \\ b-\dfrac{b}{a}\left(x-\dfrac{l}{2}\right) & \dfrac{l}{2}<x<\dfrac{l}{2}+a \\ 0 & x>\dfrac{l}{2}+a \end{cases}$$

$$y(x,t)=\sum_{n=1}^{\infty} A_n \sin(\lambda_n x)\cos(c\lambda_n t)$$

$$A_n=\frac{2}{l}\int_0^l f(x)\sin(\lambda_n x)\,dx=\frac{2}{l}\int_a^{l/2}\frac{b}{a}(x-a)\sin\lambda_n x\,dx$$

$$+\frac{2}{l}\int_{l/2}^{l/2+a}\left(b-\frac{b}{a}\left(x-\frac{l}{2}\right)\right)\sin\lambda_n x\,dx$$

```
clc
clear
L=1;
c=1;
syms x n t
a=0.25;
b=.3;
m=b/a;
y=0;
for n=1:20
 Lan=n*pi/L;
An=(2/L)*(int(m*(x-a)*sin(Lan*x),x,a,L/2)+int((b-m*(x-L/2))*sin(Lan*x),x,L/2,L/2+a));
yn=An*sin(Lan*x)*cos(c*Lan*t);
y=y+yn;
end
y=subs(y,t,0);
ezplot(y,[0 L])
```

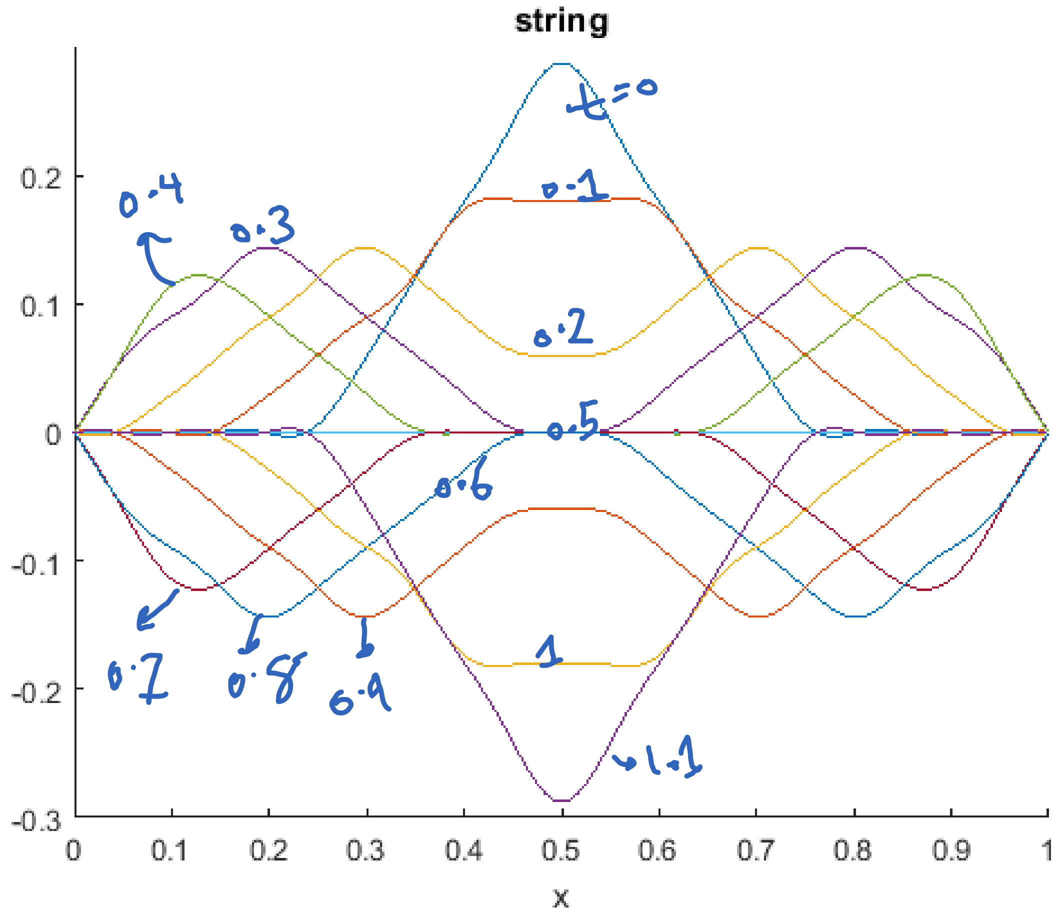


تابع $f(x)$ با تخمین

$n=20$ و $n=3$

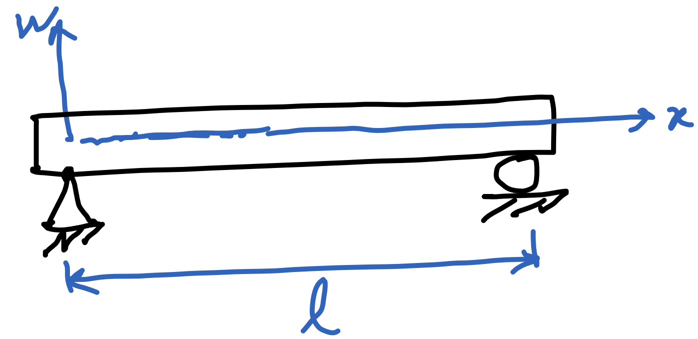
```
for tt=0:0.02:2
yp=subs(y,t,tt);
figure (1)
% hold on
ezplot(yp,[0 L -b b])
title string
pause(.1)
end
```

با توجه به شکل (۲) هم ،
لحظه‌ها موج‌های بیشتری ظاهر
می‌شوند.
۲ تا 20 مورد اول را رسم کردیم



مثال: ارتعاشات تیر متناهی با تکیه‌گاه ساده

$$EI\,W_{xxxx} + \rho A\,W_{tt} = f(x,t) \quad 0<x<L$$



BC: $w(0,t) = w(l,t) = 0$

$$EI\,W_{xx}(0,t) = EI\,W_{xx}(l,t) = 0$$

IC: $w(x,0) = g(x)$, $W_t(x,0) = h(x)$

روش جداسازی برای مسئله همگن $(f(x,t)=0)$

$$w(x,t) = X(x)T(t)$$

$$\Rightarrow EI\,X^{(4)}T + \rho A X \ddot{T} = 0$$

$$EI X^{(4)} T + \rho A X \ddot{T} = 0 \Rightarrow \frac{X^{(4)}}{X} + \frac{\rho A}{EI}\frac{\ddot{T}}{T} = 0$$

$$\Rightarrow \frac{X^{(4)}}{X} = -\frac{\rho A}{EI}\frac{\ddot{T}}{T} = \lambda^4$$

$$\Rightarrow X^{(4)} - \lambda^4 X = 0 \xrightarrow{X(x)=e^{sx}} (s^4 - \lambda^4)e^{sx} = 0 \Rightarrow s^4 = \lambda^4$$

$$\Rightarrow s^2 = \pm\lambda^2 \Rightarrow \begin{cases} s^2 = \lambda^2 \longrightarrow s_{1,2} = \pm\lambda \\ s^2 = -\lambda^2 \longrightarrow s_{3,4} = \pm\lambda i \end{cases}$$

$$\Rightarrow X(x) = A\sin\lambda x + B\cos\lambda x + C\sinh\lambda x + D\cosh\lambda x$$

$$\text{L } X(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x} + c_3 e^{\lambda x i} + c_4 e^{-\lambda x i}$$

$$X(x) = A\sin\lambda x + \cancel{B\cos\lambda x}^{\,0} + C\sinh\lambda x + \cancel{D\cosh\lambda x}^{\,0}$$

$$X'(x) = A\lambda\cos\lambda x - B\lambda\sin\lambda x + C\lambda\cosh\lambda x + D\lambda\sinh\lambda x$$

$$X''(x) = -A\lambda^2\sin\lambda x - B\lambda^2\cos\lambda x + C\lambda^2\sinh\lambda x + D\lambda^2\cosh\lambda x$$

$$W = X(0)T(t) = 0$$

$$BC: \quad w(0,t) = w(l,t) = 0 \Rightarrow X(0) = X(l) = 0$$

$$EIW_{xx}(0,t) = EIW_{xx}(l,t) = 0 \Rightarrow X''(0) = X''(l) = 0$$

$$X(0) = 0 \Rightarrow B + D = 0$$

$$X''(0) = 0 \Rightarrow -B\lambda^2 + D\lambda^2 = 0$$

$$\Rightarrow B = D = 0$$

$$X(x) = \underline{A\sin\lambda x +} B\cos\lambda x + C\sinh\lambda x + D\cosh\lambda x$$

$$X'(x) = A\lambda\cos\lambda x - B\lambda\sin\lambda x + C\lambda\cosh\lambda x + D\lambda\sinh\lambda x$$

$$X''(x) = -A\lambda^2\sin\lambda x - B\lambda^2\cos\lambda x + C\lambda^2\sinh\lambda x + D\lambda^2\cosh\lambda x$$

$$X(l) = 0 \Rightarrow A\sin\lambda l + C\sinh\lambda l = 0$$

$$X''(l) = 0 \Rightarrow -A\lambda^2\sin\lambda l + C\lambda^2\sinh\lambda l = 0$$

$$\Rightarrow \begin{cases} (1)\ A\sin\lambda l + C\sinh\lambda l = 0 \\ (2)\ -A\sin\lambda l + C\sinh\lambda l = 0 \end{cases} \Rightarrow (1)+(2) \Rightarrow 2C\sinh\lambda l = 0 \Rightarrow C = 0$$

$$\Rightarrow A\sin\lambda l = 0 \xRightarrow{A\neq 0} \sin\lambda l = 0$$

$$\Rightarrow \lambda l = n\pi \rightarrow \lambda_n = \frac{n\pi}{L}, \quad X_n(x) = \sin\lambda_n x$$

محاسبه پاسخ زمانی

$$-\overset{1/\beta^2}{\frac{\rho A}{EI}}\frac{\ddot{T}}{T} = \lambda^4 \Rightarrow \ddot{T} + \beta^2\lambda^4 T = 0$$

$$\Rightarrow T_n(t) = A\sin(\beta\lambda_n^2 t) + B\cos(\beta\lambda_n^2 t)$$

پاسخ نهایی ارتعاشات آزاد به صورت زیر خواهد شد

$$W(x,t) = \sum_{n=1}^{\infty} \sin\lambda_n x \left\{A_n \sin\beta\lambda_n^2 t + B_n\cos\beta\lambda_n^2 t\right\}$$

$$W(x,t) = \sum_{n=1}^{\infty} \sin\lambda_n x \left\{ A_n \sin\beta\lambda_n^2 t + B_n \cos\beta\lambda_n^2 t \right\}$$

با فرض $f(x,t)=0$,

$$\text{IC:} \quad W(x,0) = g(x), \quad W_t(x,0) = h(x)$$

$$W(x,0) = g(x) = \sum_{n=1}^{\infty} B_n \sin\lambda_n x \Rightarrow B_n = \frac{2}{\ell}\int_0^{\ell} g(x)\sin(\lambda_n x)\,dx$$

$$W_t(x,t) = \sum_{n=1}^{\infty} \sin\lambda_n x \left\{ A_n \beta\lambda_n^2 \cos\beta\lambda_n^2 t - B_n \beta\lambda_n^2 \sin\beta\lambda_n^2 t \right\}$$

$$\Rightarrow W_t(x,0) = h(x) = \sum_{n=1}^{\infty} \underline{A_n \beta\lambda_n^2} \sin(\lambda_n x) \Rightarrow A_n \beta\lambda_n^2 = \frac{2}{\ell}\int_0^{\ell} h(x)\sin(\lambda_n x)\,dx$$

$$W(x,t)=\sum_{n=1}^{\infty}\underline{\sin\lambda_n x}\,\overbrace{\left\{A_n\sin\beta\lambda_n^2 t+B_n\cos\beta\lambda_n^2 t\right\}}^{\eta_n(t)}$$

با فرض $f(x,t)\neq 0$ و

$$IC:\quad w(x,0)=0\ ,\quad w_t(x,0)=0$$

پاسخ ارتعاشات اجباری از روش بسط توابع ویژه بدست می‌آید

$$W(x,t)=\sum_{n=1}^{\infty}\eta_n(t)\,\underline{\sin(\lambda_n x)}$$

این پاسخ را در معادله جایگذاری می‌کنیم

$$w_{tt}+\frac{EI}{\rho A}w_{xxxx}=\frac{1}{\rho A}f(x,t)$$

همچنین باید تابع $f(x,t)$ را برحسب توابع ویژه بنویسیم

تابع $f(x,t)$ را برحسب توابع ویژه بسط می‌دهیم

$$\frac{f(x,t)}{\rho A} = \sum_{n=1}^{\infty} F(n,t) \sin \lambda_n x$$

که در آن

$$F(n,t) = \frac{2}{l} \int_0^l \frac{f(x,t)}{\rho A} \sin \lambda_n x \, dx$$

حال $w(x,t)$ و $\frac{f(x,t)}{\rho A}$ را در معادله جایگذاری می‌کنیم

$$w_{tt} + \frac{EI}{\rho A} w_{xxxx} = \frac{1}{\rho A} f(x,t)$$

$$\sum_{n=1}^{\infty} \ddot{\eta}_n(t) \sin(\lambda_n x) + \frac{EI}{\rho A} \sum_{n=1}^{\infty} \eta_n(t) \lambda_n^4 \sin \lambda_n x = \sum_{n=1}^{\infty} F(n,t) \sin \lambda_n x$$

$$\sum_{n=1}^{\infty} \ddot{\eta}_n(t)\sin(\lambda_n x) + \frac{EI}{\rho A}\sum_{n=1}^{\infty}\eta_n(t)\lambda_n^4 \sin\lambda_n x = \sum_{n=1}^{\infty} F(n,t)\sin\lambda_n x$$

$$\Rightarrow \sum_{n=1}^{\infty}\left\{\ddot{\eta}_n(t) + \frac{EI}{\rho A}\lambda_n^4\eta_n(t) - F(n,t)\right\}\sin\lambda_n(x) = 0$$

$$\Rightarrow \ddot{\eta}_n(t) + \frac{EI}{\rho A}\lambda_n^4\eta_n(t) = F(n,t)$$

حال تبدیل لاپلاس نسبت به متغیر t

$$\overset{\mathcal{L}}{\Rightarrow} s^2\eta_n(s) - s\eta(0) - \eta'(0) + \beta^2\lambda_n^4\eta_n(s) = F(n,s)$$

$$\left(\mathcal{L}\left\{\frac{\partial^2 f}{\partial t^2}\right\} = s^2F(s) - sf(0) - f'(0)\right)$$

$$\Rightarrow s^2\eta_n(s) - s\eta(0) - \eta'(0) + \beta^2\lambda_n^4\,\eta_n(s) = F(n,s)$$

$$\Rightarrow (s^2+\beta^2\lambda_n^4)\,\eta_n(s) = s\,\eta_n(0) + \eta_n'(0) + F(n,s)$$

از طرفی داشتیم که $\eta_n(t) = A_n \sin\beta\lambda_n^2 t + B_n \cos\beta\lambda_n^2 t$ پس

$$\eta_n(0) = B_n, \qquad \eta_n'(0) = A_n\beta\lambda_n^2$$

با فرض $w(x,0) = g(x) = 0,\ w_t(x,0) = h(x) = 0 \Rightarrow A_n = B_n = 0$

$$(s^2+\beta^2\lambda_n^4)\,\eta_n(s) = F(n,s) \Rightarrow \eta_n(s) = \frac{F(n,s)}{s^2+\beta^2\lambda_n^4}$$

$$\eta_n(s) = \frac{F(n,s)}{s^2+\beta^2\lambda_n^4} = \frac{F(n,s)}{s^2+a^2}$$ با فرض $\beta^2\lambda_n^4 = a^2$

$$\Rightarrow \eta_n(s) = \frac{F(n,s)}{a}\,\frac{a}{s^2+a^2}$$

حال باید از $\eta_n(s)$ عکس تبدیل لاپلاس بگیریم. $\eta(t)$ را محاسبه کنیم

$$\eta_n(t) = \mathcal{L}^{-1}\{\eta_n(s)\} = \mathcal{L}^{-1}\left\{\frac{F(n,s)}{a}\,\frac{a}{s^2+a^2}\right\}$$

از رابطه پیچش تبدیل لاپلاس داریم

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t f(\tau)\,g(t-\tau)\,d\tau$$

$$\eta_n(t) = \mathcal{L}^{-1}\{\eta_n(s)\} = \mathcal{L}^{-1}\left\{ \frac{F(n,s)}{a} \, \frac{a}{s^2+a^2} \right\}$$

$$\mathcal{L}^{-1}\{F(s)G(s)\} = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

از طرفی داریم

$$\mathcal{L}^{-1}\{F(n,s)\} = F(n,t) \, , \quad \mathcal{L}^{-1}\left\{ \frac{a}{s^2+a^2} \right\} = \sin at$$

$$\Rightarrow \eta_{n_p}(t) = \int_0^t \frac{F(n,\tau)}{a} \sin a(t-\tau)\, d\tau$$

خصوصی

حال شرایط اولیه را در نظر می گیریم

$$\eta_n(0) = B_n \,, \qquad \eta_n'(0) = A_n \beta \lambda_n^2$$

$$(s^2+\beta^2\lambda_n^4)\,\eta_n(s) = s\,\eta_n(0) + \eta_n'(0) + F(n,s)$$

$$\Rightarrow \eta_n(s) = \frac{sB_n}{s^2+a^2} + \frac{aA_n}{s^2+a^2}$$

$$\xrightarrow{\mathcal{L}^{-1}} \eta_{n_h}(t) = \mathcal{L}^{-1}\left\{\frac{B_n s}{s^2+a^2}\right\} + \mathcal{L}^{-1}\left\{\frac{A_n a}{s^2+a^2}\right\} = B_n \cos at + A_n \sin at$$

همگن

$$\eta_n(t) = \eta_{nh} + \eta_{np}$$

نهایتاً پاسخ به ازای

$$IC: \quad w(x,0)=g(x), \quad w_t(x,0)=h(x) \quad \text{و} \quad f(x,t)=0$$

$$w(x,t)=\sum_{n=1}^{\infty} \sin\lambda_n x \left\{A_n \sin\beta\lambda_n^2 t + B_n \cos\beta\lambda_n^2 t\right\}$$

که در آن

$$B_n=\frac{2}{l}\int_0^l g(x)\sin(\lambda_n x)\,dx \quad \text{و} \quad \lambda_n=\frac{n\pi}{l}$$

$$A_n\beta\lambda_n^2=\frac{2}{l}\int_0^l h(x)\sin(\lambda_n x)\,dx \quad \text{و} \quad \frac{1}{\beta}=\frac{\rho A}{EI}$$

و پاسخ ابتدایی $f(x,t) \neq 0$ و $W_t(x,0)=0$ ، $\text{IC}: \; W(x,0)=0$

$$W(x,t) = \sum_{n=1}^{\infty} \eta_n(t) \sin(\lambda_n x)$$

که در آن

$$\eta_n(t) = \int_0^t \frac{F(n,\tau)}{a} \sin a(t-\tau)\, d\tau \quad , \quad a = \beta \lambda_n^2$$

$$F(n,t) = \frac{2}{\ell} \int_0^{\ell} \frac{f(x,t)}{\rho A} \sin \lambda_n x \, dx$$

و پاسخ ابتدایی $f(x,t) \neq 0$ و

$$IC: \quad w(x,0) = g(x) \,, \quad w_t(x,0) = h(x)$$

که در آن

$$W(x,t) = \sum_{n=1}^{\infty} \eta_n(t) \sin(\lambda_n x)$$

$$\eta_n(t) = \overbrace{B_n \cos at + A_n \sin at}^{\eta_h} + \overbrace{\int_0^t \frac{F(n,\tau)}{a} \sin a(t-\tau)\, d\tau}^{\eta_p}$$

$$F(n,t) = \frac{2}{l} \int_0^l \frac{f(x,t)}{\rho A} \sin \lambda_n x \, dx \quad , \quad a = \beta \lambda_n^2$$

$$B_n = \frac{2}{l} \int_0^l g(x) \sin(\lambda_n x)\, dx \quad , \quad \lambda_n = \frac{n\pi}{l}$$

$$A_n \beta \lambda_n^2 = \frac{2}{l} \int_0^l h(x) \sin(\lambda_n x)\, dx \quad , \quad \frac{1}{\beta} = \frac{\rho A}{EI}$$

$$\mathcal{F}_s \longrightarrow \mathcal{L} \longrightarrow \mathcal{L}^{-1} \longrightarrow \mathcal{F}_s^{-1}$$

روش دوم : تبدیل فوریه سینوسی متناهی

$$\mathcal{F}_s\left\{\frac{\partial^4 f}{\partial x^4}\right\} = -\frac{2}{L}\lambda_n\left[f''(l)\cos n\pi - f''(0)\right] + \frac{2}{l}\lambda_n^3\left[f(l)\cos n\pi - f(0)\right] + \lambda_n^4 F_s(s)$$

$$EI\left\{-\frac{2}{L}\lambda_n\left[\cancelto{0}{w''(l,t)}(-1)^n - \cancelto{0}{w''(0,t)}\right] + \frac{2}{l}\lambda_n^3\left[\cancelto{0}{w(l,t)}(-1)^n - \cancelto{0}{w(0,t)}\right] + \lambda_n^4 w(n,t)\right\}$$

$$+\rho A\,\ddot{w}(n,t) = F(n,t)$$

$$\Rightarrow EI\lambda_n^4 W(n,t) + \rho A \ddot{W}(n,t) = F(n,t)$$

که در آن

$$F(n,t) = \frac{2}{l}\int_0^l f(x)\sin\lambda_n x \, dx$$

$$\beta^2 = \frac{EI}{\rho A}$$

$$\frac{1}{\rho A} \Rightarrow \ddot{W}(n,t) + \beta^2\lambda_n^4 W(n,t) = \frac{F(n,t)}{\rho A}, \quad W(n,0) = G(n), \quad \dot{W}(n,0) = H(n)$$

$$\ddot{w}(n,t) + a^2 w(n,t) = \frac{F(n,t)}{\rho A} \quad ,$$

$$w(n,0) = G(n) \quad , \quad \dot{w}(n,0) = H(n)$$

که در آن

$$G(n) = \frac{2}{l} \int_0^l g(x) \sin \lambda_n x \, dx$$

$$H(n) = \frac{2}{l} \int_0^l h(x) \sin \lambda_n x \, dx$$

$$\overset{\mathcal{L}}{\Longrightarrow} \; s^2 W(n,s) - s W(n,0) - \dot{w}(n,0) + a^2 W(n,s) = \frac{F(n,s)}{\rho A}$$

$$\Rightarrow (s^2 + a^2) W(n,s) = \frac{F(n,s)}{\rho A} + s G(n) + H(n)$$

$$(s^2+a^2)W(n,s) = \frac{F(n,s)}{\rho A} + sG(n) + H(n)$$

$$\Rightarrow W(n,s) = \frac{F(n,s)}{a\rho A}\,\frac{a}{s^2+a^2} + \frac{G(n)\,s}{s^2+a^2} + \frac{H(n)}{s^2+a^2}$$

$$\mathcal{L}^{-1} \Rightarrow w(n,t) = \frac{1}{\rho A}\int_0^t F(n,\tau)\sin(t-\tau)\,d\tau + G(n)\cos at + \frac{H(n)}{a}\sin at$$

عکس تبدیل فوریه

$$\mathcal{F}^{-1} \Rightarrow w(x,t) = \sum_{n=1}^{\infty} w(n,t)\sin \lambda_n x$$

در مقایسه جواب روش قبل، به جای $\eta_n(t)$، $W(n,t)$ داریم

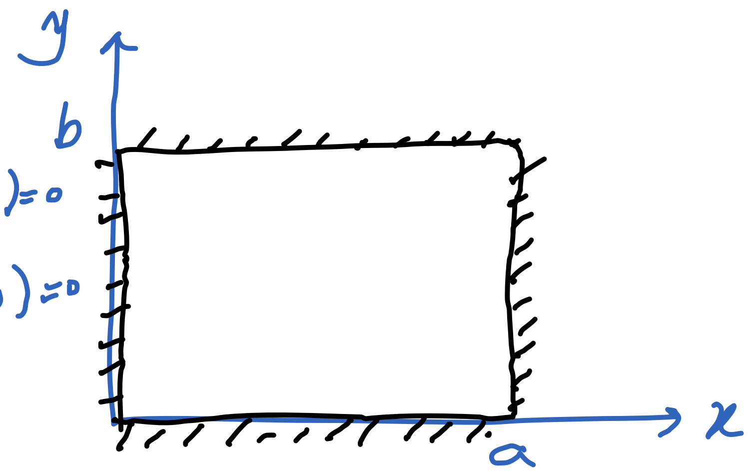
جلسه بیست و ششم

مثال: انتشار موج دو بعدی

ارتعاشات آزاد غشای مستطیلی

$$c^2(w_{xx}+w_{yy})=w_{tt}$$

$$0<x<a,\quad 0<y<b$$



$$BC:\ w(0,y,t)=w(a,y,t)=0 \rightarrow X(0)=X(a)=0$$

$$w(x,0,t)=w(x,b,t)=0 \rightarrow Y(0)=Y(b)=0$$

$$IC:\ w(x,y,0)=f(x,y)$$

$$w_t(x,y,0)=g(x,y)$$

با روش جداسازی

$$w(x,y,t)=X(x)Y(y)T(t)$$

$$c^2(X''YT + XY''T) = XY\ddot{T} \xRightarrow{\frac{1}{XYT}} \underbrace{\frac{X''}{X}}_{-\lambda^2} + \underbrace{\frac{Y''}{Y}}_{-\gamma^2} = \frac{1}{c^2}\frac{\ddot{T}}{T}$$

$$\left.\begin{array}{l} \frac{X''}{X} = -\lambda^2 \Rightarrow X'' + \lambda^2 X = 0 \\ X(0) = X(a) = 0 \end{array}\right\} \Rightarrow \boxed{X(x) = \sin\lambda_n x \,, \quad \lambda_n = \frac{n\pi}{a}} \,, \; n = 1, 2, \dots$$

$$\frac{Y''}{Y} = -\gamma^2 \longrightarrow \boxed{Y(y) = \sin\gamma_m y \,, \quad \gamma_m = \frac{m\pi}{b}} \,, \; m = 1, 2, 3, \dots$$

$$Y(0) = Y(b) = 0$$

$$\frac{X''}{X} + \frac{Y''}{Y} = \frac{1}{c^2}\frac{\ddot{T}}{T} \Rightarrow \frac{\ddot{T}}{c^2 T} = -\lambda_n^2 - \gamma_m^2$$

$$\ddot{T} + c^2(\lambda_n^2 + \gamma_m^2)T = 0 \Rightarrow \ddot{T} + \omega_{mn}^2 T = 0$$

$$\omega_{mn}^2 = c^2(\lambda_n^2 + \gamma_m^2) \quad \sqrt{}$$

$$\Rightarrow T(t) = A_{mn}\cos\omega_{mn}t + B_{mn}\sin\omega_{mn}t$$

نهایتاً جواب کلی می‌شود

$$w(x,y,t)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} X_n(x)\,Y_m(y)\,T_{mn}(t)$$

$$=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[A_{mn}\cos\omega_{mn}t+B_{mn}\sin\omega_{mn}t\right]\sin\lambda_n x\,\sin\gamma_m y$$

ضرایب A_{mn} و B_{mn} با اعمال شرایط اولیه بدست می‌آیند

$$\Rightarrow w(x,y,0)=f(x,y)=\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}\sin\lambda_n x\,\sin\gamma_m y$$

پس A_{mn} را از رابطه سری فوریه دوگانه سینوسی محاسبه کرد.

$$A_{mn}=\frac{2}{a}\,\frac{2}{b}\int_0^a\!\!\int_0^b f(x,y)\,\sin\lambda_n x\,\sin\gamma_m y\,dx\,dy$$

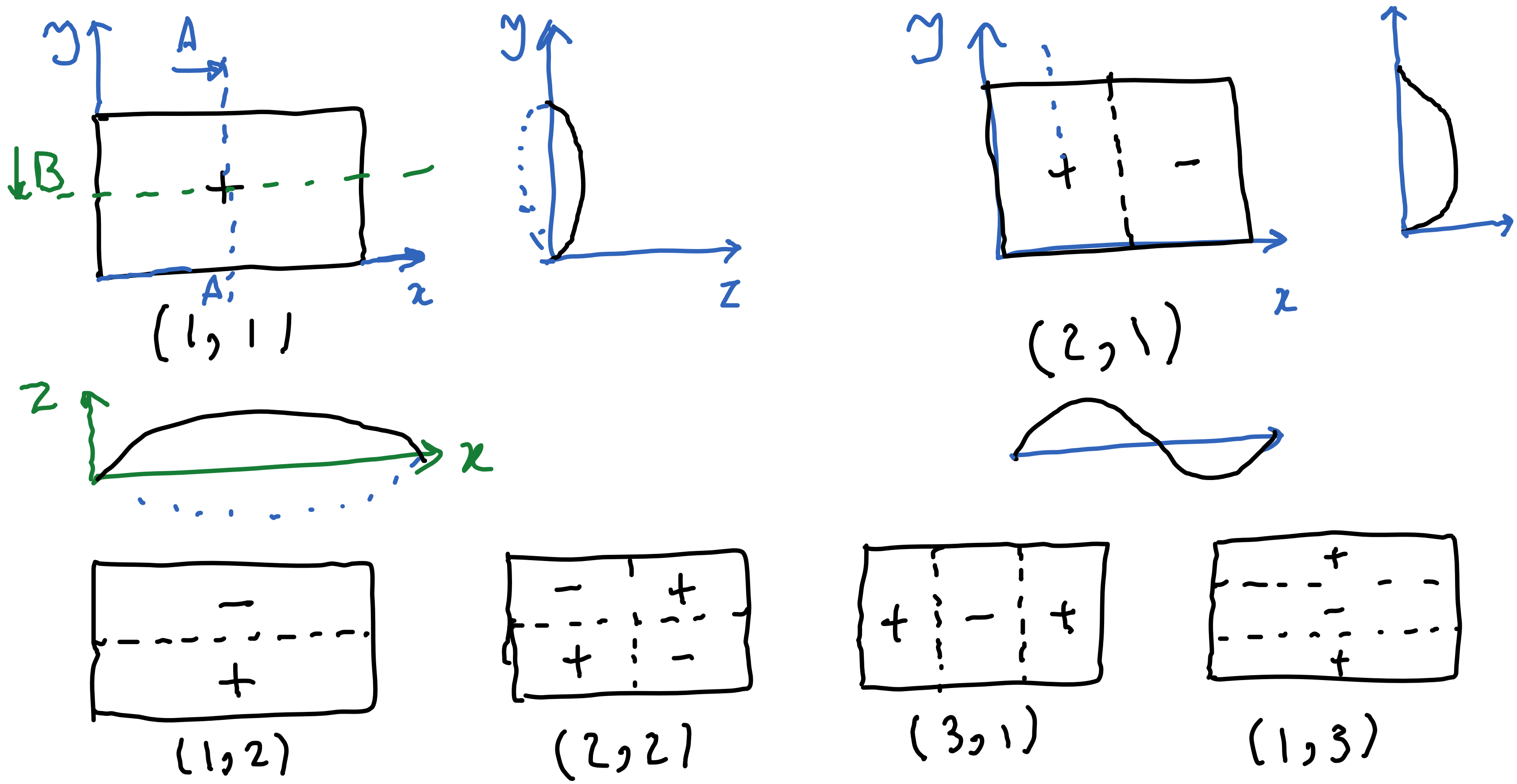
همچنین با اعمال شرط اولیه

$$w_t(x,y,0) = g(x,y)$$

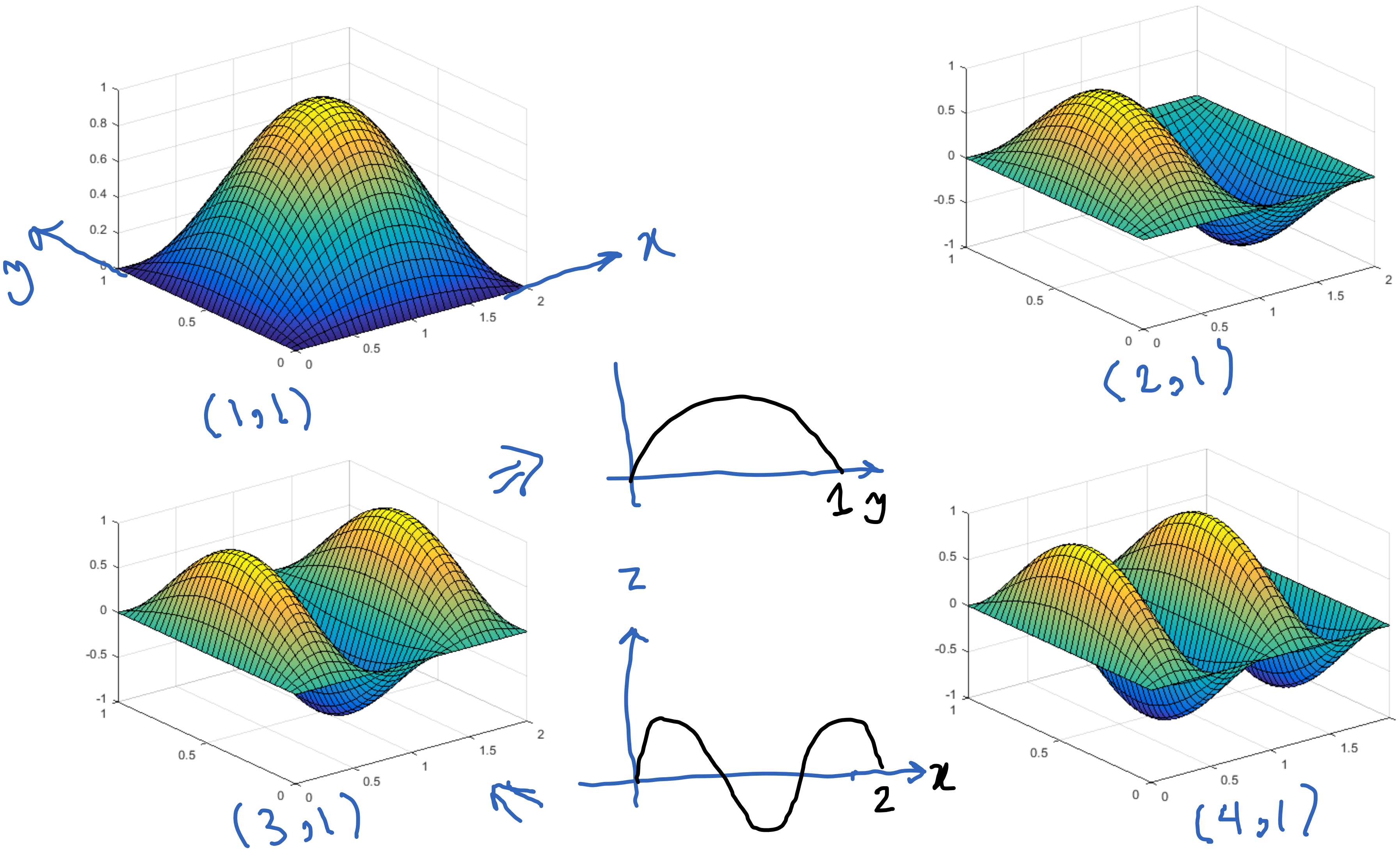
$$g(x,y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \underline{B_{mn}\,\omega_{mn}} \sin \lambda_n x \sin \gamma_m y$$

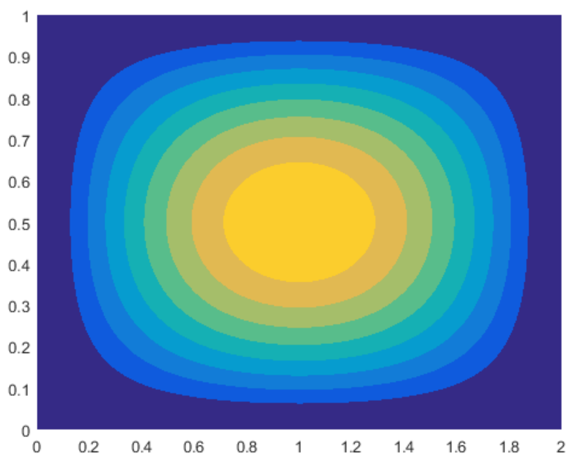
$$\Rightarrow B_{mn}\,\omega_{mn} = \frac{2}{a}\,\frac{2}{b} \int_0^a \int_0^b g(x,y) \sin \lambda_n x \sin \gamma_m y \, dx\, dy$$

شکل مود (n,m) با توابع $\left\{\sin\frac{n\pi x}{a} , \sin\frac{m\pi y}{b}\right\}$ نمایش داده می‌شود.

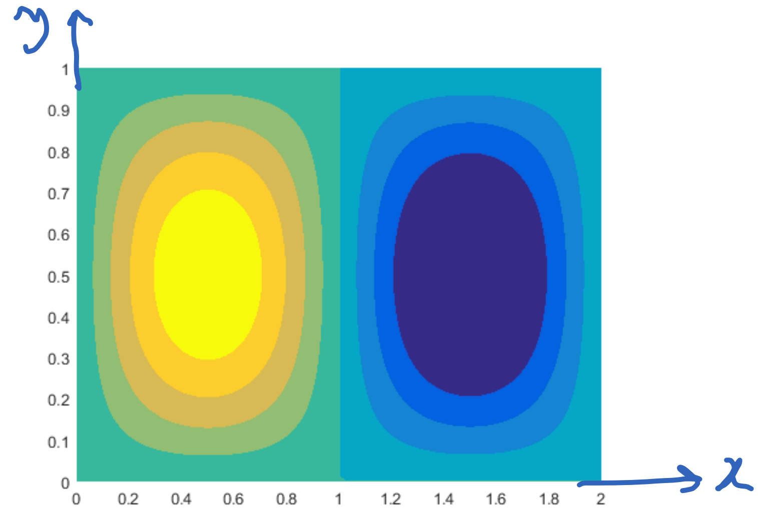


```
clc
clear

syms y x
n=1;m=2;
a=2;
b=1;
Lan=n*pi/a;
Gam=m*pi/b;

figure (1)
% fcontour(@(x,y) sin(Lan*x)*sin(Gam*y),[0 a 0 b],'Fill','on')
fsurf(@(x,y) sin(Lan*x)*sin(Gam*y),[0 a 0 b])
```

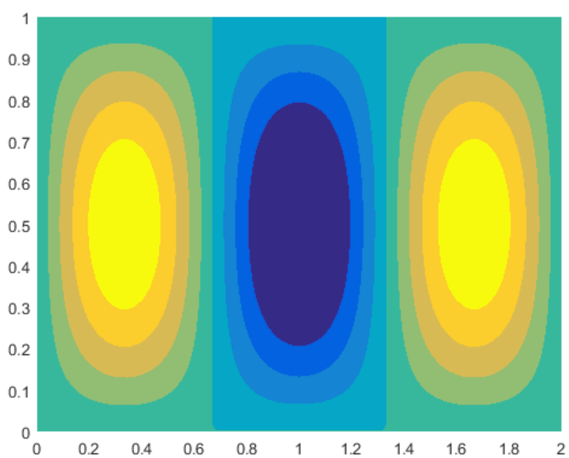

x
y
(1و1)
(2و1)
(3و1)
(4و1)
1 y
z
2 x


0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
0 0.2 0.4 0.6 0.8 1 1.2 1.4 1.6 1.8 2

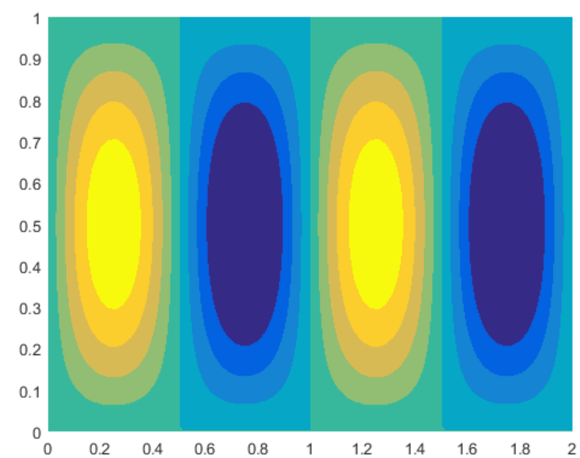
(1,1)


y
0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
0 0.2 0.4 0.6 0.8 1 1.2 1.4 1.6 1.8 2
x

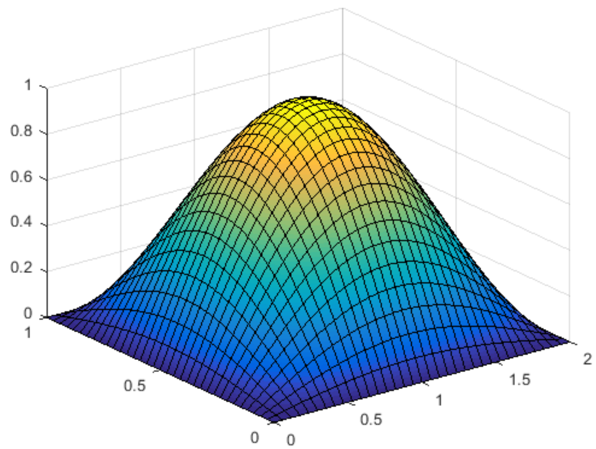
(2,1)


0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
0 0.2 0.4 0.6 0.8 1 1.2 1.4 1.6 1.8 2

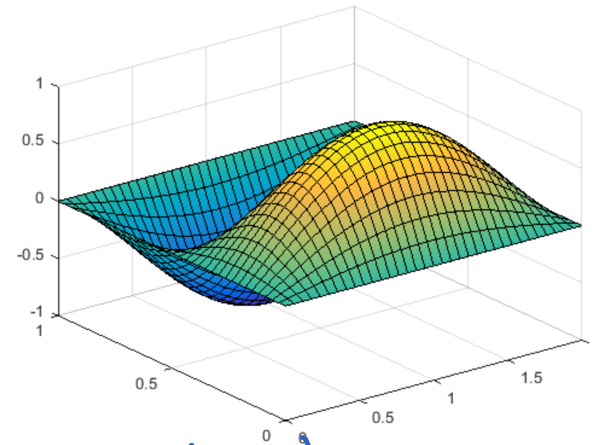
(3,1)


0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8 0.9 1
0 0.2 0.4 0.6 0.8 1 1.2 1.4 1.6 1.8 2

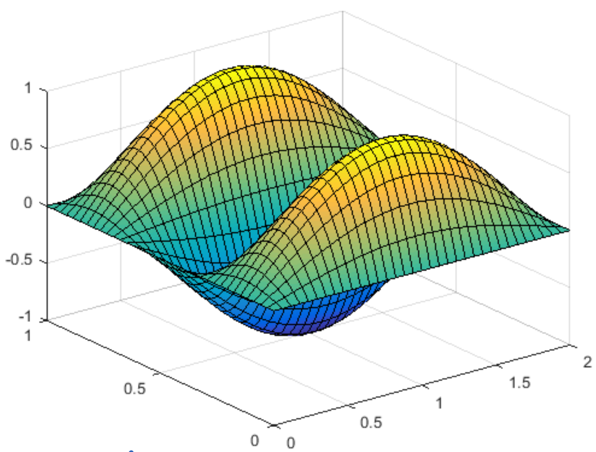
(4,1)


1
0.8
0.6
0.4
0.2
0
1
0.5
0
0
0.5
1
1.5
2

(1,1)


1
0.5
0
-0.5
-1
1
0.5
0
0
0.5
1
1.5
2

(1,2)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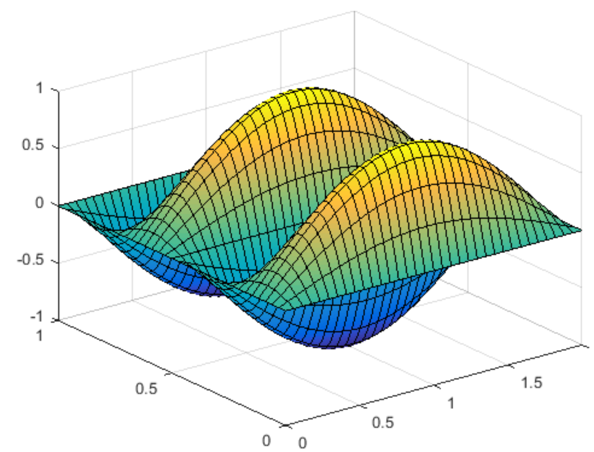
0
-0.5
-1
1
0.5
0
0
0.5
1
1.5
2

(1,3)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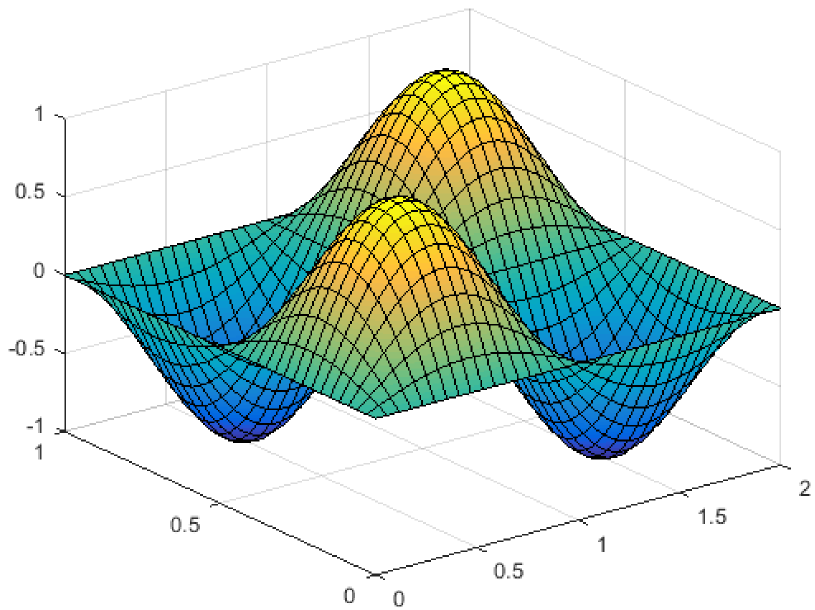
0.5
0
-0.5
-1
1
0.5
0
0
0.5
1
1.5
2

(1,4)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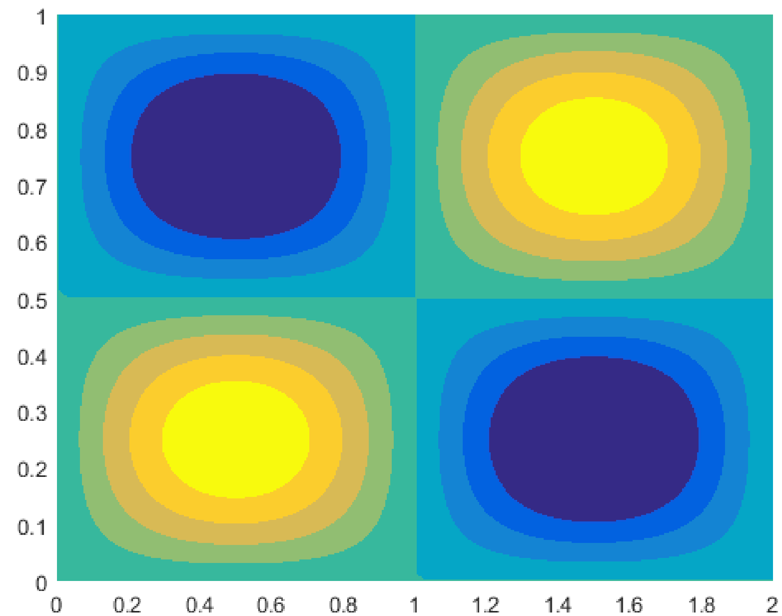
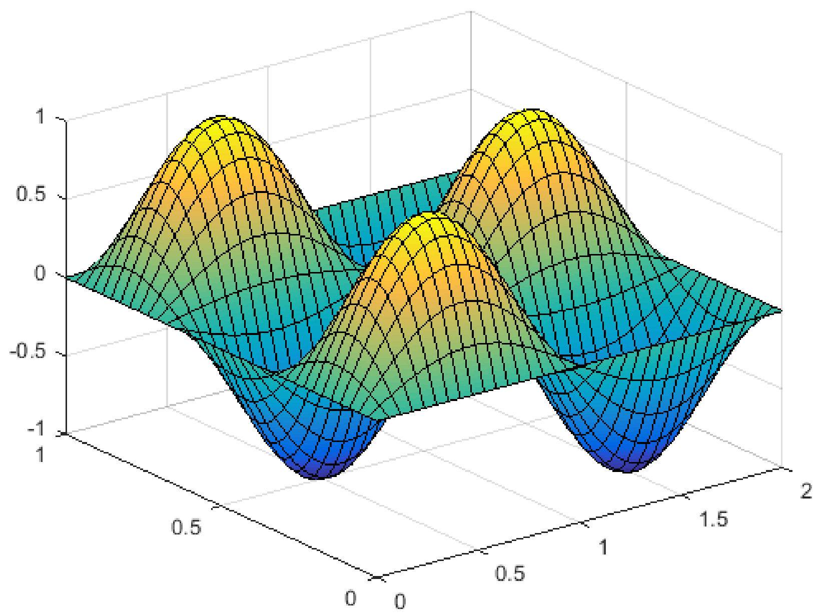
0.5
0
-0.5
-1
1
0.5
0
0
0.5
1
1.5
2

(2,2)


1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2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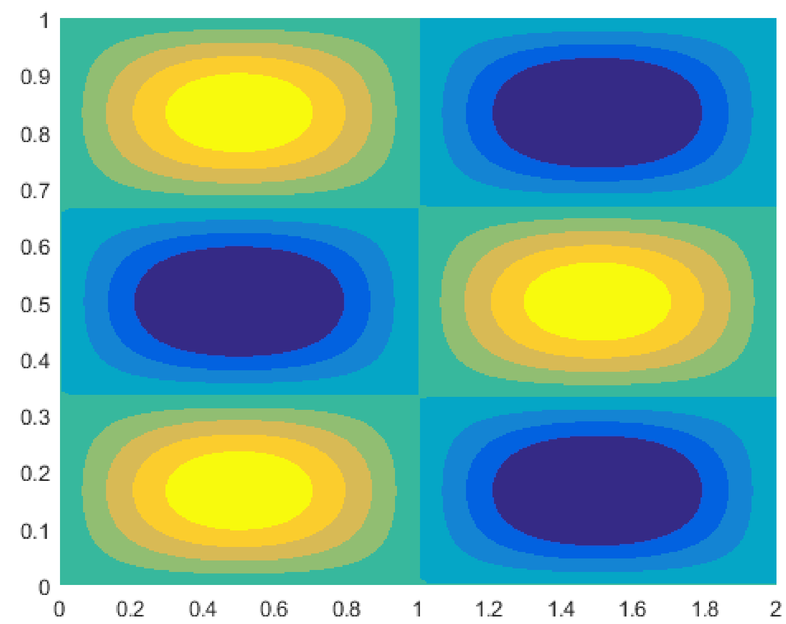
0.5
0
-0.5
-1
1
0.5
0
0
0.5
1
1.5
2


1
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0.7
0.6
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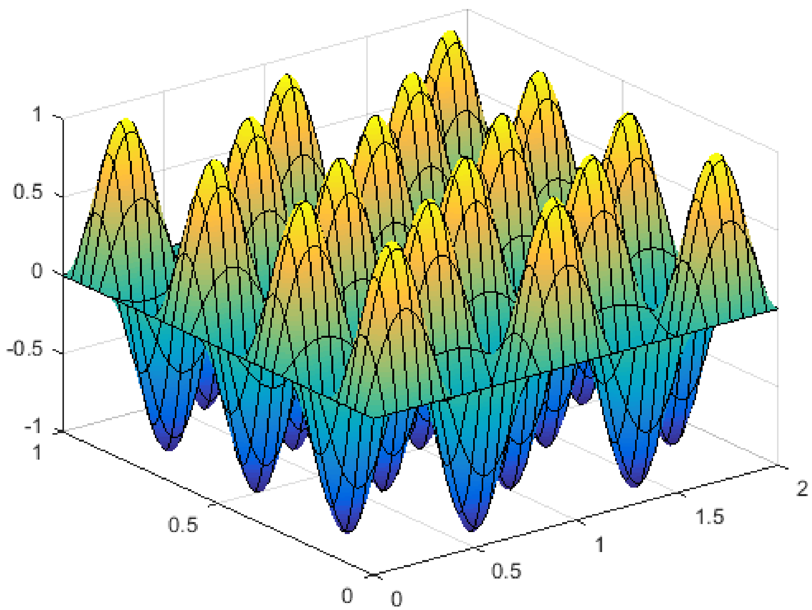
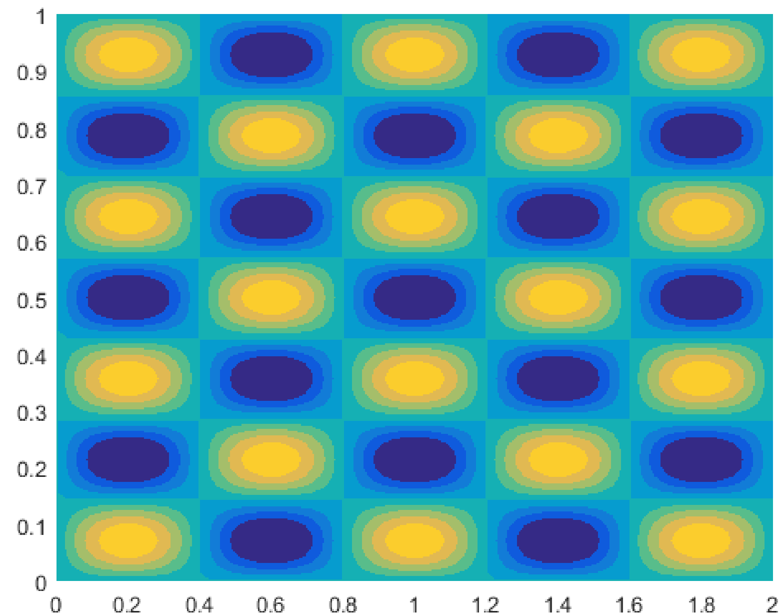
0.3
0.2
0.1
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

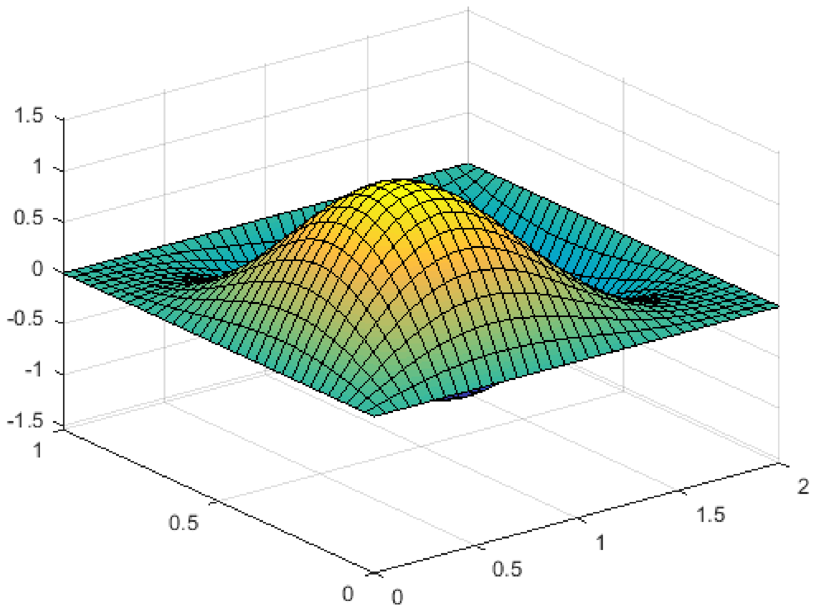
(2,3)

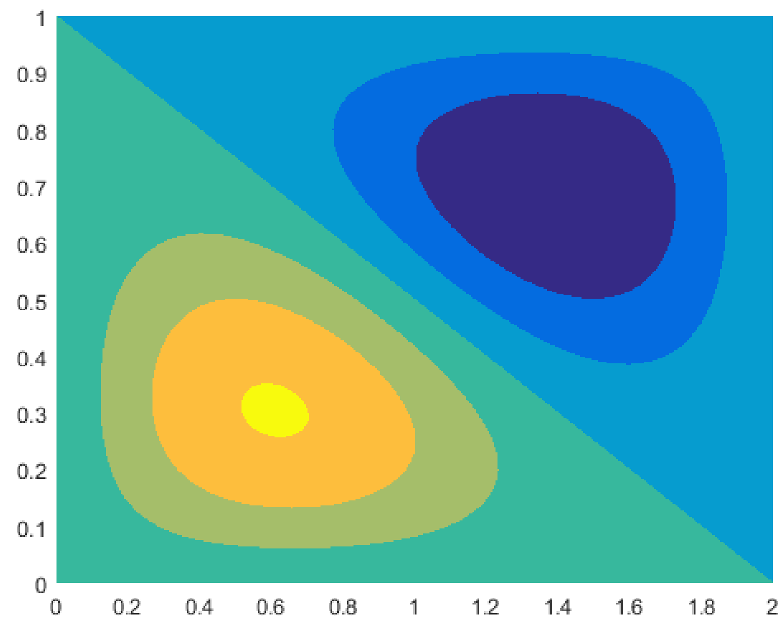

1
0.5
0
-0.5
-1
1
0.5
0
0
0.5
1
1.5
2


1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2

(5,7)

$$\sin\frac{\pi x}{a}\,\sin\frac{2\pi y}{b}+\sin\frac{2\pi x}{a}\,\sin\frac{\pi y}{b}$$





$(1,2) + (2,1)$ حاصل جمع دو شکل مود مختلف

Forced vibration of a semi-infinite string

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + \underline{f(t)}, \quad 0 < x < \infty, \quad t > 0$$

BC: $u(0,t) = 0$

IC: $u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = 0$

یک روش می‌تواند تبدیل فوریه سینوسی نیم‌متناهی باشد

روش دیگر استفاده از تبدیل لاپلاس ✓

$$\mathcal{L}\{u(x,t)\} = U(x,s)$$

$$s^2 U(x,s) - s\,\overset{0}{u(x,0)} - \overset{0}{u_t(x,0)} = c^2 U''(x,s) + F(s)$$

$$\Rightarrow -c^2 U''(x,s) + s^2 U(x,s) = F(s)$$

که تابع نسبت به x می‌باشد

$$-c^2 U''(x,s) + s^2 U(x,s) = F(s)$$

$$\Rightarrow -U'' + \frac{s^2}{c^2} U = \frac{F(s)}{c^2}, \qquad -\frac{\partial^2 U(x,s)}{\partial x^2} + \frac{s^2}{c^2} U(x,s) = \frac{F(s)}{c^2}$$

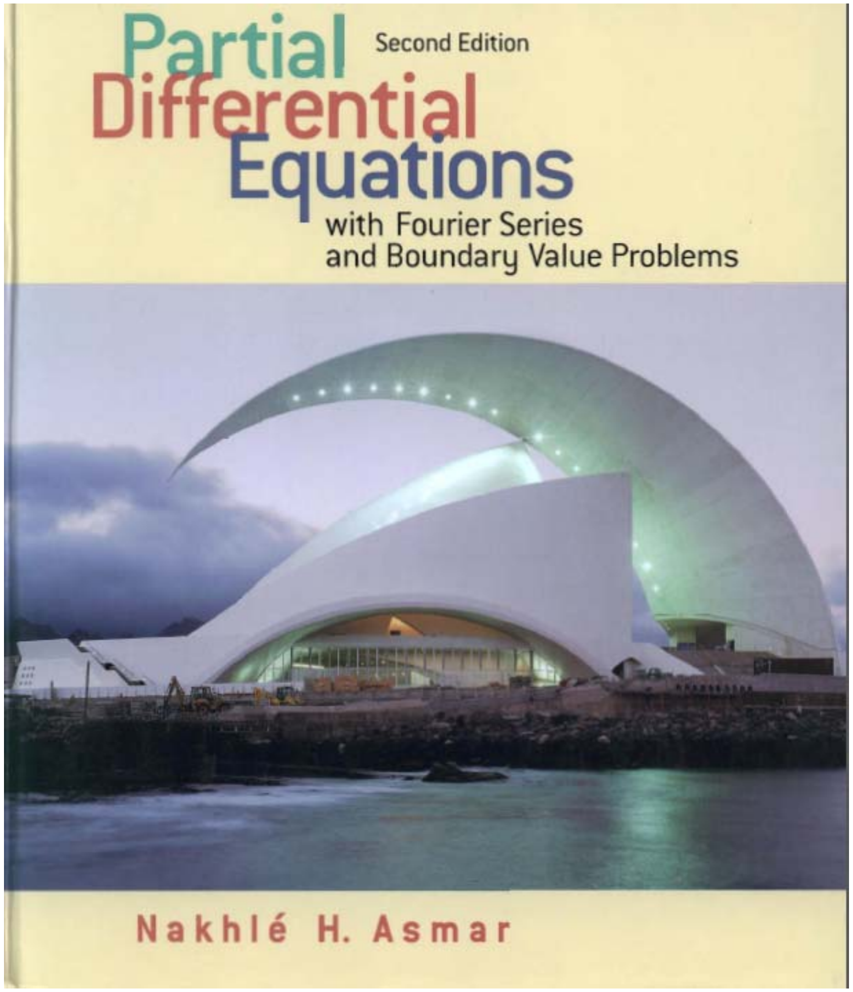
$$U(x,s) = \underbrace{A(s) e^{-\frac{s}{c}x} + B(s) e^{\frac{s}{c}x}}_{U_h} + \underbrace{\frac{F(s)}{s^2}}_{U_p} = U_h + U_p$$

$$x \to \infty \Rightarrow e^{\frac{s}{c}x} \to \infty \Rightarrow B(s) = 0$$

$$\Rightarrow U(x,s) = A(s) e^{-\frac{s}{c}x} + \frac{F(s)}{s^2}$$

$$\text{From BC}, \; U(0,s) = 0 \Rightarrow A(s) = -\frac{F(s)}{s^2}$$

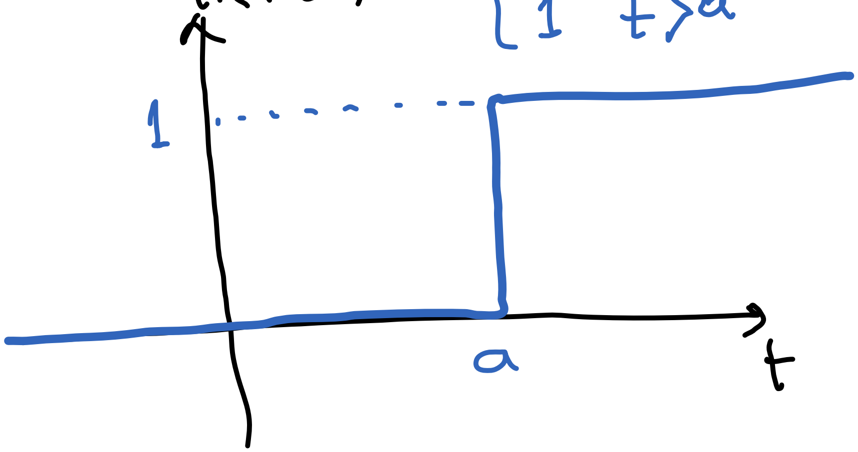
$$\Rightarrow U(x,s) = \frac{-F(s) e^{-\frac{s}{c}x}}{s^2} + \frac{F(s)}{s^2} = \frac{F(s)}{s^2}\left(1 - e^{-\frac{s}{c}x}\right)$$



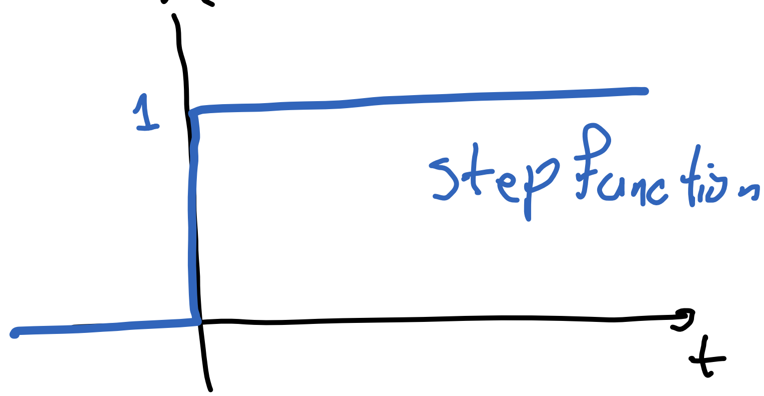
جداول پیوست آن خیلی کامل و کاربردی است،

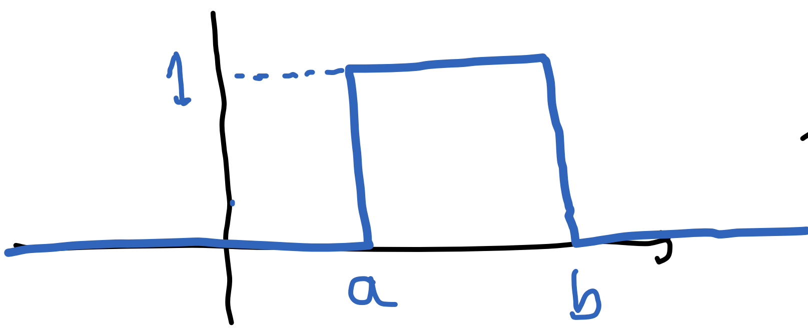
Heaviside step function $H(t-a)$, or $u(t-a)$

$$H(t-a) = \begin{cases} 0 & t<a \\ 1 & t>a \end{cases}$$



$H(t)$, $a=0$



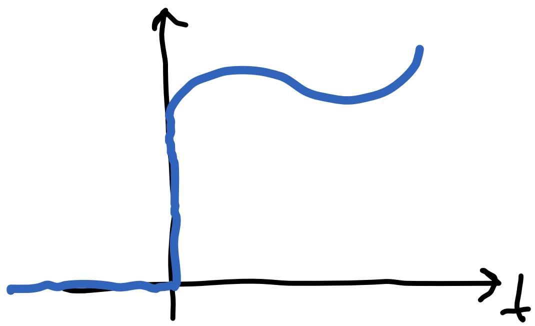


$\rightarrow H(t-a) - H(t-b)$

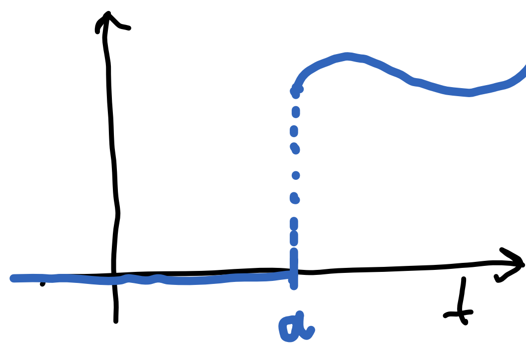
$t>0$

تابع تأخیردار شده

$f(t)$

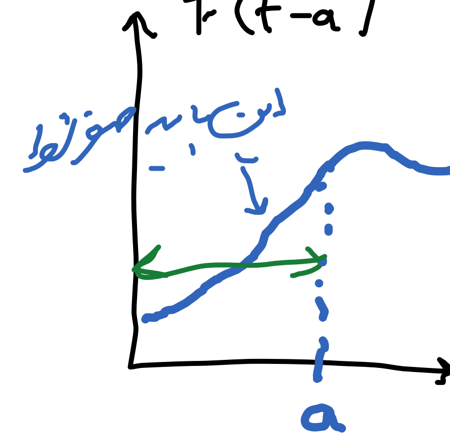


$f(t-a)\,H(t-a)$



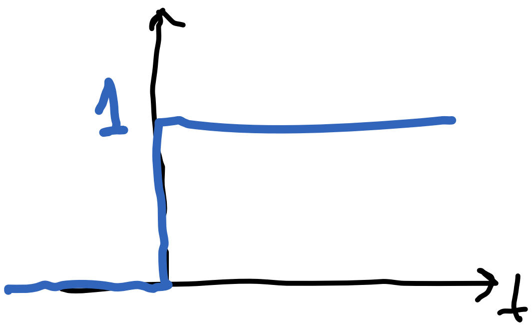
،

$f(t-a)$

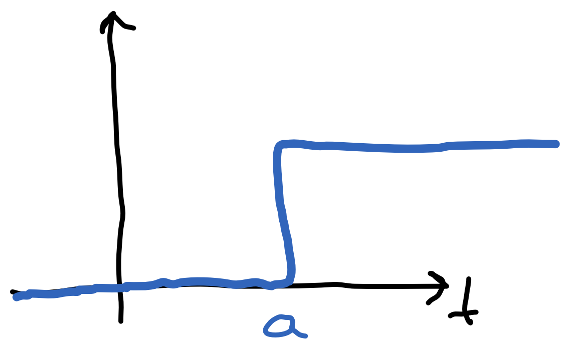


$$\mathcal{L}\left[f(t)\right]=F(s)\quad ,\quad \mathcal{L}\left[f(t-a)\,H(t-a)\right]=e^{-as}F(s)$$

$f(t) = 1$



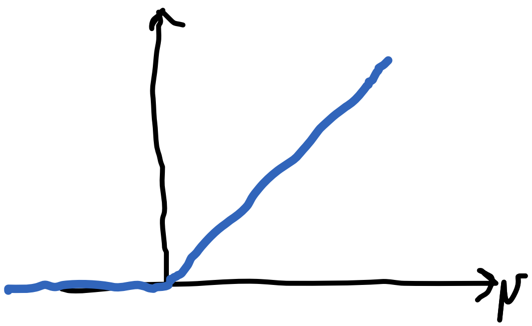
$1\, H(t-a)$



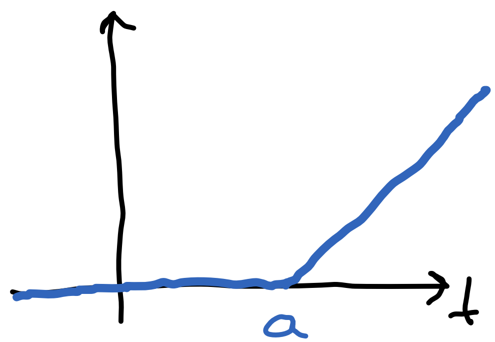
$F(s) = \frac{1}{s}$

$\frac{e^{-as}}{s}$

$f(t) = t$



$(t-a)\,H(t-a)$



$$F(s) = \frac{1}{s^2}$$

$$\frac{e^{-as}}{s^2}$$

$$\mathcal{L}[f(t)] = F(s) \longrightarrow \mathcal{L}[f(t-a)H(t-a)] = e^{-as}F(s)$$

$$u(x,s) = F(s)\underbrace{\frac{1-e^{-\frac{s}{c}x}}{s^2}}_{G(s)} \quad , \quad G(s) = \frac{1-e^{-\frac{x}{c}s}}{s^2} = \frac{1}{s^2} - \frac{e^{-\frac{x}{c}s}}{s^2}$$

$$u = F(s)\,G(s)$$

$$\mathcal{L}^{-1}\left[\frac{1}{s^2}\right] = t$$ از جدول داریم

$$\Rightarrow g(t) = \mathcal{L}^{-1}[G(s)] = t - \left(t-\frac{x}{c}\right)H\left(t-\frac{x}{c}\right)$$

$$\mathcal{L}^{-1}[F(s)] = f(t)$$

$$V(x,s) = F(s)\,G(s)$$

از رابطه پیچش برای تبدیل لاپلاس،

$$\mathcal{L}^{-1}\left[F(s)G(s)\right] = \int_0^t f(\tau)\,g(t-\tau)\,d\tau = \int_0^t f(t-\tau)\,g(\tau)\,d\tau$$

$$V(x,s) = F(s)\,\frac{1-e^{-\frac{s}{c}x}}{s^2} \quad \text{و} \quad g(t) = \mathcal{L}^{-1}\left[G(s)\right] = t - \left(t-\frac{x}{c}\right)H\left(t-\frac{x}{c}\right)$$

$$\Rightarrow u(x,t) = \int_0^t f(t-\tau)\left[\tau - \left(\tau-\frac{x}{c}\right)H\left(\tau-\frac{x}{c}\right)\right]d\tau$$

vibration of string with gravitational acc.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \underbrace{- g}_{f(t)} ,\qquad f(t) = -g \Rightarrow f(t-\tau) = -g$$

$$u(x,t) = \int_0^t f(t-\tau)\left[\tau - \left(\tau - \frac{x}{c}\right) H\left(\tau - \frac{x}{c}\right)\right] d\tau$$

$$\Rightarrow u(x,t) = -g \int_0^t \left[\tau - \left(\tau - \frac{x}{c}\right) H\left(\tau - \frac{x}{c}\right)\right] d\tau$$

$$= -g \int_0^t \tau \, d\tau + g \int_0^t \left(\tau - \frac{x}{c}\right) H\left(\tau - \frac{x}{c}\right) d\tau$$

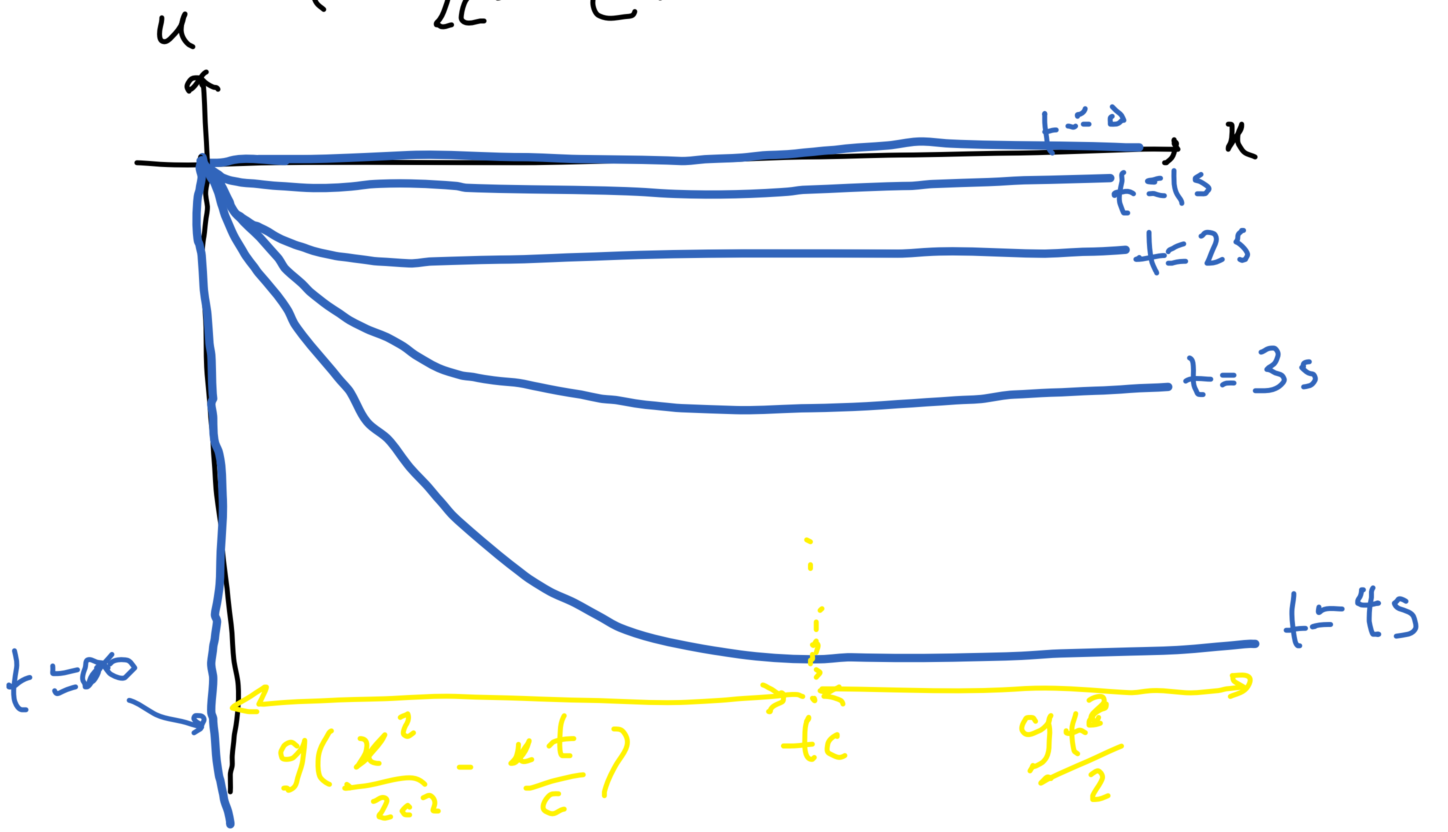
$$u(x,t) = -g\int_0^t \tau\, d\tau + g\int_0^t \left(\tau - \frac{x}{c}\right) H\left(\tau - \frac{x}{c}\right) d\tau$$

$$= \begin{cases} -g\int_0^t \tau\, d\tau & t < \frac{x}{c} \\ -g\int_0^t \tau\, d\tau + g\int_{\frac{x}{c}}^t \left(\tau - \frac{x}{c}\right) d\tau, & t > \frac{x}{c} \end{cases}$$

$$x > tc$$

$$= \begin{cases} -g\left[\frac{\tau^2}{2}\right]_0^t = -\frac{gt^2}{2} \\ -\frac{gt^2}{2} + g\left[\frac{\tau^2}{2} - \frac{x}{c}\tau\right]_{x/c}^t = -\frac{gt^2}{2} + g\left[\frac{t^2}{2} - \frac{x}{c}t - \frac{x^2}{2c^2} + \frac{x^2}{c^2}\right] \end{cases}$$

$$u(x,t) = \begin{cases} -\dfrac{gt^2}{2} & x > tc \\ g\left(\dfrac{x^2}{2c^2} - \dfrac{xt}{c}\right) & x < tc \end{cases}$$



مثال: ارتعاشات اجباری سیم نامتناهی با شرایط مرزی و اولیه غیرهمگن

مثال ۴-۹

$$y_{tt} = c^2 y_{xx} \quad , \quad 0 < x < \infty \quad , \quad t > 0$$

$$BC: \quad y(0,t) = f_1(t)$$

$$IC: \quad y(x,0) = f(x) \quad , \quad y_t(x,0) = g(x)$$

با اعمال تبدیل فوریه سینوسی نیمه متناهی

$$F_s\{y(x,t)\} = Y(w,t)$$

$$\ddot{Y}(w,t) + c^2 w^2 Y(w,t) = \frac{2c^2}{\pi} w f_1(t)$$

$$IC: \quad Y(w,0) = F(w) \quad , \quad \dot{Y}(w,0) = G(w)$$

$$\ddot{y}(w,t) + c^2 w^2 y(w,t) = \frac{2c^2}{\pi} w f_1(t)$$

IC: $y(w,0) = F(w)$, $\dot{y}(w,0) = G(w)$

باگرفتن تبدیل لاپلاس $\mathcal{L}\{y(w,t)\} = Y(w,s)$

$$s^2 Y(w,s) - sY(w,0) - \dot{Y}(w,0) + c^2 w^2 Y(w,s) = \frac{2c^2}{\pi} w F_1(s)$$

$$\Rightarrow Y(w,s)\left[s^2 + c^2 w^2\right] = \frac{2c^2}{\pi} w F_1(s) + sF(w) + G(w)$$

$$\Rightarrow Y(w,s) = \frac{2c^2}{\pi} F_1(s) \frac{w}{s^2 + c^2 w^2} + \frac{s F(w)}{s^2 + c^2 w^2} + \frac{G(w)}{s^2 + c^2 w^2}$$

$$Y(w,s) = \frac{2c^2}{M} F_1(s) \frac{w}{s^2+c^2w^2} + \frac{s\,F(w)}{s^2+c^2w^2} + \frac{G(w)}{s^2+c^2w^2}$$

حال عکس تبدیل لاپلاس می‌گیریم

$$s \longrightarrow t$$

$$Y(w,s) = F(w)\frac{s}{s^2+c^2w^2} + \frac{G(w)}{cw}\frac{cw}{s^2+c^2w^2} + \frac{2c}{M}F_1(s)\frac{cw}{s^2+c^2w^2}$$

$$\mathcal{L}^{-1} \Rightarrow y(w,t) = F(w)\cos(cwt) + \frac{G(w)}{cw}\sin(cwt) + \frac{2c}{M}\int_0^t f_1(\tau)\sin cw(t-\tau)\,d\tau$$

$$\mathcal{L}^{-1}\left[F_1(s)\,G(s)\right] = \int_0^t f_1(\tau)\,g(t-\tau)\,d\tau, \quad g(t) = \mathcal{L}^{-1}\left[G(s)\right] = \sin cwt$$

$$y(w,t) = \overbrace{F(w)\cos(cwt)}^{\text{ترم اول}} + \overbrace{\frac{G(w)}{cw}\sin(cwt)}^{\text{ترم دوم}} + \frac{2c}{\pi}\int_0^t f_1(\tau)\sin cw(t-\tau)\,d\tau$$

حال عکس تبدیل فوریه سینوسی،

$$\mathcal{F}_s^{-1}\left[F(w)\right] = f(x) = \int_0^\infty F(w)\sin wx\,dw \quad , \quad w \to x$$

برای ترم اول

$$\mathcal{F}_s^{-1}\left\{F(w)\cos cwt\right\} = \int_0^\infty F(w)\cos cwt \sin wx\,dw$$

$$= \frac{1}{2}\int_0^\infty F(w)\left\{\sin w(x-ct) + \sin w(x+ct)\right\}dw = \frac{1}{2}\left\{f(x-ct) + f(x+ct)\right\}$$

عکس تبدیل فوریه سینوسی سری دوم

$$F_s^{-1}\left\{G(w)\,\frac{\sin(cwt)}{cw}\right\}$$

$$F_{\cancel{s}c}^{-1}\left\{\frac{\sin at}{a}\right\} = ?$$

از کتاب Haberman

$\frac{1}{\pi}\int_0^\infty f(\bar{x})[g(x-\bar{x})-g(x+\bar{x})]\,d\bar{x}$ $=\frac{1}{\pi}\int_0^\infty g(\bar{x})[f(x+\bar{x})-f(\bar{x}-x)]\,d\bar{x}$	$S[f(x)]C[g(x)]$	Convolution (Exercise 10.5.6)

رابطه بالا در صورتی صادق است که $f(x)$ یک تابع فرد باشد و $g(x)$ یک تابع زوج باشد

https://www.sciencedirect.com/topics/mathematics/fourier-sine-transform

$$x_o(t) = \frac{t}{|t|}x(|t|) \quad \text{and} \quad y_o(t) = \frac{t}{|t|}y(|t|)$$

$\frac{t}{|t|} = \text{sign}(t)$

are convolved. The FCT of the convolution reduces to the product of the FSTs of the two functions $x(t)$ and $y(t)$,

$$2\pi X_s(\omega)Y_s(\omega) = F_c\left\{\int_0^\infty x(\tau)[y(t+\tau) + y_o(t-\tau)]d\tau\right\}. \quad (2.37)$$

$\rightarrow \int_0^\infty x(\tau)\left[y(t+\tau) + \text{sign}(t-\tau)y(t-\tau)\right]d\tau$

(2.38a)

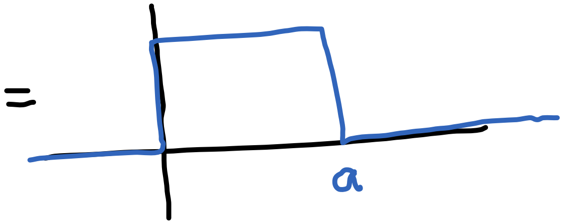
$$2\pi X_s(\omega)Y_c(\omega) = F_s\left\{\int_0^\infty x(\tau)[y(|t-\tau|) - y(t+\tau)]d\tau\right\},$$

or

(2.38b)

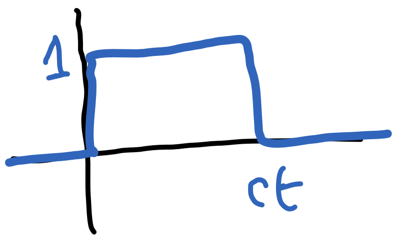
$$2\pi X_s(\omega)Y_c(\omega) = F_S\left\{\int_0^\infty y(\tau)[x(t+\tau) + x_o(t-\tau)]d\tau\right\}.$$

$$F_c^{-1}\left\{\frac{2}{\pi}\,\frac{\sin a\omega}{\omega}\right\} = f(x) =$$



$$= H(x) - H(x-a)$$

$$F_c^{-1}\left\{\frac{2}{\pi}\,\frac{\sin ct\omega}{\omega}\right\} =$$



$$= H(x) - H(x-ct)$$

$$F_s^{-1}\left\{G(\omega)\,\frac{\sin c\omega t}{c\omega}\right\} = F_s^{-1}\left\{\underbrace{\frac{\pi}{2c}\,G(\omega)}_{X_s}\;\underbrace{\frac{2\sin(ct\omega)}{\pi\;\omega}}_{Y_c}\right\}$$

Table of Fourier Cosine Transforms

$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \mathcal{F}_c(f)(\omega) \cos \omega x \, d\omega,$ $0 < x < \infty$	$\mathcal{F}_c(f)(\omega) = \widehat{f}_c(\omega) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \cos \omega x \, dx,$ $0 \leq \omega < \infty$
1. $\begin{cases} 1 & \text{if } 0 < x < a \\ 0 & \text{otherwise} \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin a\omega}{\omega}$
2. $e^{-ax}, \quad a > 0$	$\sqrt{\frac{2}{\pi}} \frac{a}{a^2+\omega^2}$
3. $x e^{-ax}, \quad a > 0$	$\sqrt{\frac{2}{\pi}} \frac{a^2-\omega^2}{(a^2+\omega^2)^2}$
4. $e^{-a x^2/2}, \quad a > 0$	$\frac{1}{\sqrt{a}} e^{-\omega^2/2a}$
5. $\cos ax \, e^{-ax}, \quad a > 0$	$\sqrt{\frac{2}{\pi}} \frac{a\omega^2+2a^3}{4a^4+\omega^4}$
6. $\sin ax \, e^{-ax}, \quad a > 0$	$\sqrt{\frac{2}{\pi}} \frac{2a^3-a\omega^2}{4a^4+\omega^4}$
7. $\frac{a}{a^2+x^2}, \quad a > 0$	$\sqrt{\frac{\pi}{2}} e^{-a\omega}$
8. $x^p, \quad 0 < p < 1$	$\sqrt{\frac{2}{\pi}} \frac{\Gamma(p)\cos(p\omega/2)}{\omega^p}$
9. $\begin{cases} \cos x & \text{if } 0 < x < a \\ 0 & \text{otherwise} \end{cases}$	$\frac{1}{\sqrt{2\pi}} \left[\frac{\sin a(1-\omega)}{1-\omega} + \frac{\sin a(1+\omega)}{1+\omega} \right]$

$f(x)=\int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}\,d\omega$	$F(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx$	Reference
$e^{-\alpha x^2}$	$\frac{1}{\sqrt{4\pi\alpha}}e^{-\omega^2/4\alpha}$	Gaussian (Sec. 10.3.3)
$\sqrt{\frac{\pi}{\beta}}e^{-x^2/4\beta}$	$e^{-\beta\omega^2}$	Gaussian (Sec. 10.3.3)
$\frac{\partial f}{\partial t}$	$\frac{\partial F}{\partial t}$	Derivatives (Sec. 10.4.2)
$\frac{\partial f}{\partial x}$	$-i\omega F(\omega)$	Derivatives (Sec. 10.4.2)
$\frac{\partial^2 f}{\partial x^2}$	$(-i\omega)^2 F(\omega)$	Derivatives (Sec. 10.4.2)
$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\overline{x})g(x-\overline{x})d\overline{x}$	$F(\omega)G(\omega)$	Convolution (Sec. 10.4.3)
$\delta(x-x_0)$	$\frac{1}{2\pi}e^{i\omega x_0}$	Dirac delta function (Exercise 10.3.18)
$f(x-\beta)$	$e^{i\omega\beta}F(\omega)$	Shifting theorem (Exercise 10.3.5)
$xf(x)$	$-i\frac{dF}{d\omega}$	Multiplication by x (Exercise 10.3.8)
$\frac{2\alpha}{x^2+\alpha^2}$	$e^{-\lvert\omega\rvert\alpha}$	Exercise 10.3.7
$f(x)=\begin{cases} 0 & \lvert x\rvert > a \\ 1 & \lvert x\rvert < a \end{cases}$	$\frac{1}{\pi}\frac{\sin a\omega}{\omega}$	Exercise 10.3.6

Table 10.4.1: Fourier Transform

Table 10.5.2: Fourier Cosine Transform

$f(x) = \int_0^\infty F(\omega)\cos\omega x \, d\omega$	$C[f(x)] = F(\omega) = \frac{2}{\pi}\int_0^\infty f(x)\cos\omega x \, dx$	Reference
$\frac{df}{dx}$	$-\frac{2}{\pi}f(0) + \omega S[f(x)]$	Derivatives (Sec. 10.5.4)
$\frac{d^2 f}{dx^2}$	$-\frac{2}{\pi}\frac{df}{dx}(0) - \omega^2 F(\omega)$	
$\frac{\beta}{x^2+\beta^2}$	$e^{-\omega\beta}$	Exercise 10.5.1
$e^{-\epsilon x}$	$\frac{2}{\pi}\cdot\frac{\epsilon}{\epsilon^2+\omega^2}$	Exercise 10.5.2
$e^{-\alpha x^2}$	$2\frac{1}{\sqrt{4\pi\alpha}}e^{-\omega^2/4\alpha}$	Exercise 10.5.3
$\frac{1}{\pi}\int_0^\infty g(\bar{x})[f(x-\bar{x})+f(x+\bar{x})]d\bar{x}$	$F(\omega)G(\omega)$	Convolution (Exercise 10.5.7)

Table 10.5.1: Fourier Sine Transform

$f(x) = \int_0^\infty F(\omega) \sin \omega x \, d\omega$	$S[f(x)] = F(\omega) = \frac{2}{\pi}\int_0^\infty f(x) \sin \omega x \, dx$	Reference
$\frac{df}{dx}$	$-\omega C[f(x)]$	Derivatives (Sec. 10.5.4)
$\frac{d^2 f}{dx^2}$	$\frac{2}{\pi}\omega f(0) - \omega^2 F(\omega)$	
$\frac{x}{x^2+\beta^2}$	$e^{-\omega\beta}$	Exercise 10.5.1
$e^{-\epsilon x}$	$\frac{2}{\pi} \cdot \frac{\omega}{\epsilon^2+\omega^2}$	Exercise 10.5.2
1	$\frac{2}{\pi} \cdot \frac{1}{\omega}$	Exercise 10.5.9
$\frac{1}{\pi}\int_0^\infty f(\overline{x})[g(x-\overline{x}) - g(x+\overline{x})]d\overline{x}$ $= \frac{1}{\pi}\int_0^\infty g(\overline{x})[f(x+\overline{x}) - f(\overline{x}-x)]d\overline{x}$	$S[f(x)]C[g(x)]$	Convolution (Exercise 10.5.6)

$$F_s^{-1}\left\{G(\omega)\frac{\sin c\omega t}{c\omega}\right\} = F_s^{-1}\left\{\frac{\pi}{2c}G(\omega)\,\frac{2\sin(ct\omega)}{\pi\,\omega}\right\}$$

$t \to x$

$\tau \to u$

X_s, G_s

Y_c, F_c

$f_c(x)$

$$2\pi X_s(\omega)Y_c(\omega) = F_S\left\{\int_0^\infty y(\tau)[x(t+\tau)+x_o(t-\tau)]d\tau\right\}. \quad (2.38b)$$

$$F_c^{-1}\left\{\frac{2\sin(ct\omega)}{\pi\omega}\right\} = H(x)-H(x-ct) =$$

$$F_s^{-1}\left\{\frac{\pi}{2c}G(\omega)\right\} = \frac{\pi}{2c}\,g(x)$$

$$= \frac{1}{\pi}\frac{\pi}{2c}\int_0^\infty [H(u)-H(u-ct)]\{g(x+u)+\operatorname{sign}(x-u)\,g(x-u)\}\,du$$

$$= \frac{1}{2c}\int_0^{ct}\{g(x+u)+\operatorname{sign}(x-u)\,g(x-u)\}\,du$$

ترم سوم

$$F_s^{-1}\left\{\frac{2c}{\pi}\int_0^t f_1(\tau)\sin c\omega(t-\tau)\,d\tau\right\} ,$$

$$F_s^{-1}\left[F(\omega)\right] = f(x) = \int_0^\infty F(\omega)\sin\omega x\,d\omega$$

$\Rightarrow$ ترم سوم $= \dfrac{2c}{\pi}\displaystyle\int_0^\infty\int_0^t f_1(\tau)\sin c\omega(t-\tau)\sin\omega x\,d\tau\,d\omega$

$$= \frac{2c}{\pi}\int_0^\infty\left\{\int_0^t f_1(t-\tau)\sin c\omega\tau\,d\tau\right\}\sin\omega x\,d\omega$$

با فرض $v = c\tau$

$$= \frac{2c}{\pi}\int_0^{\infty}\left\{\int_0^{t} f_1(t-\tau)\,\sin c\omega\tau\, d\tau\right\}\sin\omega x\, d\omega$$

با فرض $v = c\tau \Rightarrow d\tau = \frac{dv}{c}$

$$= \frac{2}{\pi}\int_0^{\infty}\left\{\int_0^{ct} f_1\left(t-\frac{v}{c}\right)\sin\omega v\, dv\right\}\sin\omega x\, d\omega$$

از طرفی برای تبدیل فوریه سینوسی تابع ناصفر را بدست داریم

$$\mathcal{F}_s\left\{f_1\left(t-\frac{v}{c}\right)H\left(t-\frac{v}{c}\right)\right\} = \frac{2}{\pi}\int_0^{\infty} f_1\left(t-\frac{v}{c}\right)H\left(t-\frac{v}{c}\right)\sin\omega v\, dv$$

$$= \frac{2}{\pi}\int_0^{ct} f_1\left(t-\frac{v}{c}\right)\sin\omega v\, dv$$

$$\mathcal{L}_s^{-1}\left\{\frac{2c}{m}\int_0^t f_1(\tau)\,\sin c\omega(t-\tau)\,d\tau\right\} = f_1\left(t-\frac{x}{c}\right)H\left(t-\frac{x}{c}\right)$$

پس نتیجه

$$u(x,t) = \frac{1}{2}\left[f(x-ct)+f(x+ct)\right]$$
$$+\frac{1}{2c}\int_0^{ct}\left\{g(x+u)+\operatorname{sign}(x-u)\,g(x-u)\right\}du$$
$$+f_1\left(t-\frac{x}{c}\right)H\left(t-\frac{x}{c}\right)$$

مقادیر و توابع ویژه برای شرایط مختلف مرزی دیریکله و نیومن - بازه متناهی

معادله $X'' + \lambda X = 0$ در بازه متناهی

حالت	شرایط مرزی	مقدار ویژه ($\lambda_n = {\alpha_n}^2$)	تابع ویژه	دوره تناوب تابع ویژه
۱	$X(0) = 0; X(l) = 0$ $0 \le x \le l$	$\alpha_n = \frac{n\pi}{l}\ ; n = 1,2,\dots$	$\phi_n = \sin(\alpha_n x)$	$2l$
۲	$X'(0) = 0; X'(l) = 0$ $0 \le x \le l$	$\alpha_n = \frac{n\pi}{l}\ ; n = 0,1,\dots$	$\phi_n = \cos(\alpha_n x)$	$2l$
۳	$X(0) = 0; X'(l) = 0$ $0 \le x \le l$	$\alpha_n = \frac{(2n-1)\pi}{2l}\ ; n = 1,2,\dots$	$\phi_n = \sin(\alpha_n x)$	$4l$
۴	$X'(0) = 0; X(l) = 0$ $0 \le x \le l$	$\alpha_n = \frac{(2n-1)\pi}{2l}\ ; n = 1,2,\dots$	$\phi_n = \cos(\alpha_n x)$	$4l$
۶	$X(-l) = X(l); X'(-l) = X'(l)$ $-l \le x \le l$	$\alpha_n = \frac{n\pi}{l}\ ; n = 0,1,\dots$	$\begin{cases} \phi_n = \sin(\alpha_n x)\ ; n \neq 0 \\ \phi_n = \cos(\alpha_n x) \end{cases}$	$2l$

مقادیر و توابع ویژه برای شرایط مختلف مرزی دیریکله و نیومن - بازه نامتناهی

8

معادله $X'' + \lambda X = 0$ در بازه نامتناهی

حالت	شرایط مرزی	مقدار ویژه ($\lambda = a^2$)	تابع ویژه
۱	$X(0) = 0; \lvert X(\infty)\rvert < M$ $0 \leq x < \infty$	$\forall\, \alpha > 0$	$\phi = \sin(\alpha x)$
۲	$X'(0) = 0; \lvert X(\infty)\rvert < M$ $0 \leq x < \infty$	$\forall\, \alpha \geq 0$	$\phi = \cos(\alpha x)$
۳	$\lvert X(\pm\infty)\rvert < M$ $-\infty < x < \infty$	$\forall\, \alpha \geq 0$	$\begin{cases} \phi = \sin(\alpha x)\,;\, \alpha \neq 0 \\ \phi = \cos(\alpha x) \end{cases}$