



دینامیک پیشرفته

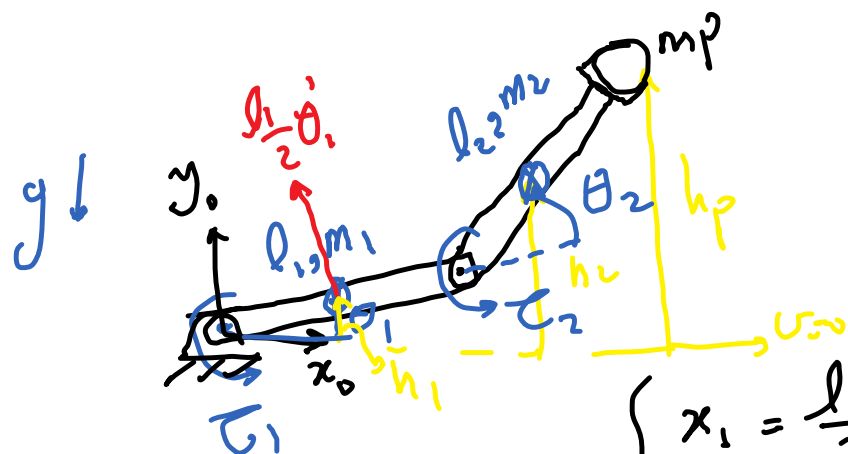
دینامیک لاگرانژین

استخراج معادلات دینامیکی ربات دولینکی به روش لاگرانژ
به همراه شبیه سازی در MATLAB

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درجه ۱ با درجه ۲ نیوتن اولیه

$$T = T_1 + T_2 + T_{mp}$$

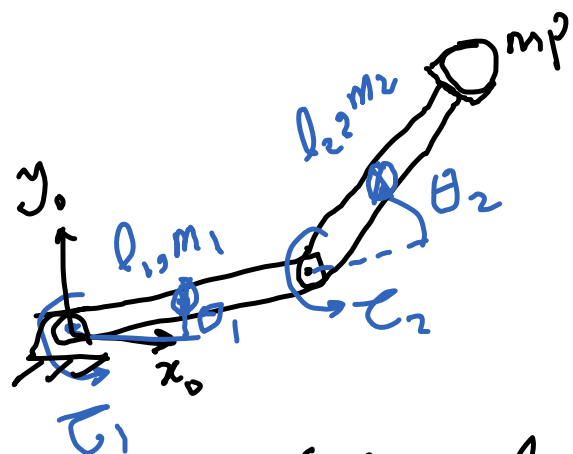
$$U = U_1 + U_2 + U_p$$

نسبت اول

$$\begin{cases} x_1 = \frac{l_1}{2} \cos \theta_1 \Rightarrow \dot{x}_1 = -\frac{l_1}{2} \dot{\theta}_1 \sin \theta_1 \\ y_1 = \frac{l_1}{2} \sin \theta_1 \Rightarrow \dot{y}_1 = \frac{l_1}{2} \dot{\theta}_1 \cos \theta_1 \end{cases}$$

$$\omega_1 = \dot{\theta}_1$$

$$v_1^2 = (\dot{x}_1^2 + \dot{y}_1^2) = \left(\frac{l_1}{2} \dot{\theta}_1 \right)^2 \quad (1)$$

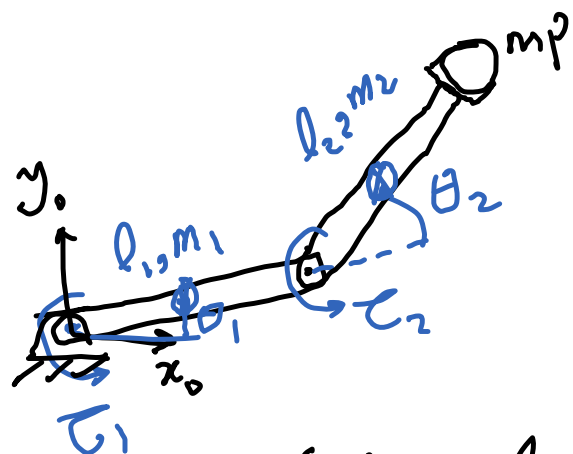


$$\begin{cases} x_2 = l_1 \cos \theta_1 + \frac{l_2}{2} \cos(\theta_1 + \theta_2) \\ y_2 = l_1 \sin \theta_1 + \frac{l_2}{2} \sin(\theta_1 + \theta_2) \end{cases}$$

$$\begin{cases} \dot{x}_2 = -l_1 \dot{\theta}_1 \sin \theta_1 - \frac{l_2}{2} (\dot{\theta}_1 + \dot{\theta}_2) \sin(\theta_1 + \theta_2) \\ \dot{y}_2 = +l_1 \dot{\theta}_1 \cos \theta_1 + \frac{l_2}{2} (\dot{\theta}_1 + \dot{\theta}_2) \cos(\theta_1 + \theta_2) \end{cases} \quad (2)$$

$$\omega_2 = \dot{\theta}_1 + \dot{\theta}_2$$

$$v_2^2 = \dot{x}_2^2 + \dot{y}_2^2 = \dots$$



$$\begin{cases} x_p = l_1 \cos \theta_1 + l_2 \cos(\theta_1 + \theta_2) \\ y_p = l_1 \sin \theta_1 + l_2 \sin(\theta_1 + \theta_2) \end{cases}$$

$$\begin{cases} \dot{x}_p = -l_1 \dot{\theta}_1 \sin \theta_1 - l_2 (\dot{\theta}_1 + \dot{\theta}_2) \sin(\theta_1 + \theta_2) \\ \dot{y}_p = +l_1 \dot{\theta}_1 \cos \theta_1 + l_2 (\dot{\theta}_1 + \dot{\theta}_2) \cos(\theta_1 + \theta_2) \end{cases} \quad (3)$$

$$v_p^2 = \dot{x}_p^2 + \dot{y}_p^2$$



$$\bar{T}_i = \frac{1}{2} m_i \dot{\mathbf{r}}_i^T \dot{\mathbf{r}}_i + \frac{1}{2} \omega_i^T \mathbf{I} \omega_i$$

مثال صفحه ۱۱

محاسبه انرژی جنبشی

$$T_1 = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} \mathbf{I}_{11} \omega_1^2 = \frac{1}{2} m_1 (\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2} \mathbf{I}_1 \dot{\theta}_1^2$$

$$T_2 = \frac{1}{2} m_2 v_2^2 + \frac{1}{2} \mathbf{I}_{22} \omega_2^2 = \frac{1}{2} m_2 (\dot{x}_2^2 + \dot{y}_2^2) + \frac{1}{2} \mathbf{I}_2 (\dot{\theta}_1 + \dot{\theta}_2)^2$$

$$T_{mp} = \frac{1}{2} m_p v_p^2 = \frac{1}{2} m_p (\dot{x}_p^2 + \dot{y}_p^2)$$

$$\mathbf{I}_1 = m_1 \frac{l_1^2}{12}, \quad \mathbf{I}_2 = m_2 \frac{l_2^2}{12}$$

مسابه انرژی پتانسیل

$$U = U_1 + U_2 + U_p$$

$$U_1 = m_1 g h_1 = m_1 g l_1 \sin \theta_1$$

$$U_2 = m_2 g h_2 = m_2 g \left(l_1 \sin \theta_1 + \frac{l_2}{2} \sin(\theta_1 + \theta_2) \right)$$

$$U_p = m_p g h_p = m_p g \left(l_1 \sin \theta_1 + l_2 \sin(\theta_1 + \theta_2) \right)$$



$$L = T - U, \quad L(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$$

$$\left\{ \begin{array}{l} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_1} \right) - \frac{\partial L}{\partial \theta_1} = \tau_1 \\ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_2} \right) - \frac{\partial L}{\partial \theta_2} = \tau_2 \end{array} \right. \quad \left| \quad \begin{array}{l} \frac{\partial L}{\partial \dot{\theta}_1} = L_{d1}(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) \\ \frac{d}{dt} L_{d1} = \frac{\partial L_{d1}}{\partial \theta_1} \dot{\theta}_1 + \frac{\partial L_{d1}}{\partial \theta_2} \dot{\theta}_2 \\ \quad + \frac{\partial L_{d1}}{\partial \dot{\theta}_1} \ddot{\theta}_1 + \frac{\partial L_{d1}}{\partial \dot{\theta}_2} \ddot{\theta}_2 \end{array} \right.$$

$$\frac{d}{dt} L_{d1} = \left[\frac{\partial L_{d1}}{\partial \theta_1} \quad \frac{\partial L_{d1}}{\partial \theta_2} \quad \frac{\partial L_{d1}}{\partial \dot{\theta}_1} \quad \frac{\partial L_{d1}}{\partial \dot{\theta}_2} \right] \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} =$$

$$P(q_1, q_2, q_3)$$

$$\frac{d}{dt} = \frac{\partial P}{\partial q_1} \dot{q}_1 + \frac{\partial P}{\partial q_2} \dot{q}_2 + \frac{\partial P}{\partial q_3} \dot{q}_3 = \begin{bmatrix} \frac{\partial P}{\partial q_1} & \frac{\partial P}{\partial q_2} & \frac{\partial P}{\partial q_3} \end{bmatrix} \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}$$

$$= \text{jacobian}(P, [q]) \dot{q}$$

$$q = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \end{bmatrix}, \quad \dot{q} = \begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \end{bmatrix}$$

$$M(q) \ddot{q} + h(q, \dot{q}) = T$$

$$\ddot{q} = M^{-1}(q) [T - h]$$

$$\left\{ \begin{aligned} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_1} \right) - \frac{\partial L}{\partial \theta_1} &= \tau_1 \end{aligned} \right.$$

$$\left\{ \begin{aligned} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_2} \right) - \frac{\partial L}{\partial \theta_2} &= \tau_2 \end{aligned} \right.$$

$$\frac{\partial L}{\partial \dot{\theta}_1} = L_{d1} = L_{d1}(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$$

$$\frac{\partial L}{\partial \dot{\theta}_2} = L_{d2} = L_{d2}(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2)$$

در حالت کلی

$$\begin{cases} \frac{\partial L}{\partial \dot{\theta}_1} = L_{d1} = L_{d1}(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) \\ \frac{\partial L}{\partial \dot{\theta}_2} = L_{d2} = L_{d2}(\theta_1, \theta_2, \dot{\theta}_1, \dot{\theta}_2) \end{cases}$$

$$\Rightarrow \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_i} \right) = \text{jacobian}(L_{di}, [\theta_1, \theta_2]) \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + \text{jacobian}(L_{di}, [\dot{\theta}_1, \dot{\theta}_2]) \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}$$

$$\frac{\partial L}{\partial \dot{\theta}} = \begin{bmatrix} \frac{\partial L}{\partial \dot{\theta}_1} \\ \frac{\partial L}{\partial \dot{\theta}_2} \end{bmatrix} \Rightarrow \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}} \right) = J_1 \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + M \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}$$



$$\begin{cases} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_1} \right) - \frac{\partial L}{\partial \theta_1} = \tau_1 \\ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_2} \right) - \frac{\partial L}{\partial \theta_2} = \tau_2 \end{cases} \quad \begin{matrix} \text{عقد (برقی)} \\ (=) \end{matrix} \quad \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = T \quad (c)$$

$$J_1 = \text{jacobian} \left(\frac{\partial L}{\partial \dot{q}}, q \right)$$

$$M = \text{jacobian} \left(\frac{\partial L}{\partial \dot{q}}, \dot{q} \right) \quad (e)$$

$$J_1 \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} + M \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} - \frac{\partial L}{\partial q} = T$$

$$\Rightarrow \ddot{q} = M^{-1} \left[\underbrace{T}_b + \underbrace{\frac{\partial L}{\partial q}}_a - \underbrace{J_1 \dot{q}}_{(d)} \right]$$



نزدیک کردن مدارات در ویم میانی حالت

$$\begin{cases} x_1 = \theta_1 \\ x_2 = \theta_2 \\ x_3 = \dot{\theta}_1 \\ x_4 = \dot{\theta}_2 \end{cases}$$

$$\Rightarrow \begin{cases} \dot{x}_1 = x_3 \\ \dot{x}_2 = x_4 \\ \begin{bmatrix} \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = M^{-1} (T - h) \end{cases}$$

فونکشن مستقیم

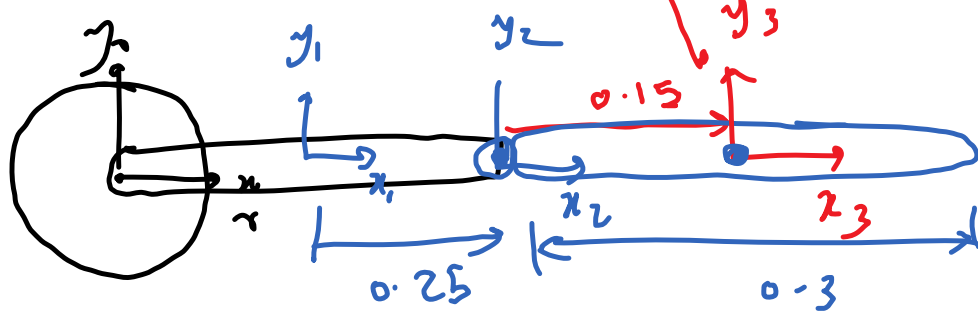
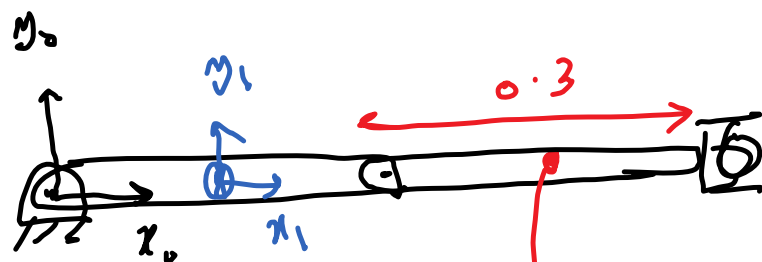
$$T \checkmark \xrightarrow{\quad} q, \dot{q}, \ddot{q}$$

دینامیک معکوس

حل مدارات دینامیک مستقیم

ODE45 :-

$g \downarrow$



$$g = \begin{bmatrix} 0 & -9.81 & 0 \end{bmatrix}$$

$$L_1 = 0.5$$

$$L_2 = 0.3$$



دینامیک پیشرفته، دینامیک لاگرانژین

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